Introduction to Feedback Control

Karl-Erik Årzen, Anton Cervin

Session outline

- **Feedback and feedforward**
- PID Control
- State-space models
- Transfer function models
- Control design using pole placement
- State feedback and observers

The Idea of Feedback

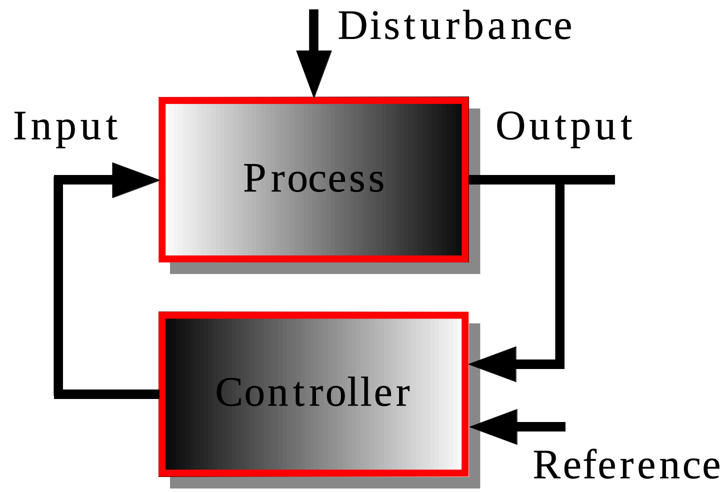
- Feedback:
  - Compare the actual result with the desired result.
  - Take actions based on the difference.
- A seemingly simple idea that is temendously powerful.
- Use of feedback has often been revolutionary.
- Feedback is also called close loop control.
- The opposite is feedforward or open loop control: make a plan and execute it.
- Feedback and feedforward are key ideas ideas in the discipline of control.

Automatic control

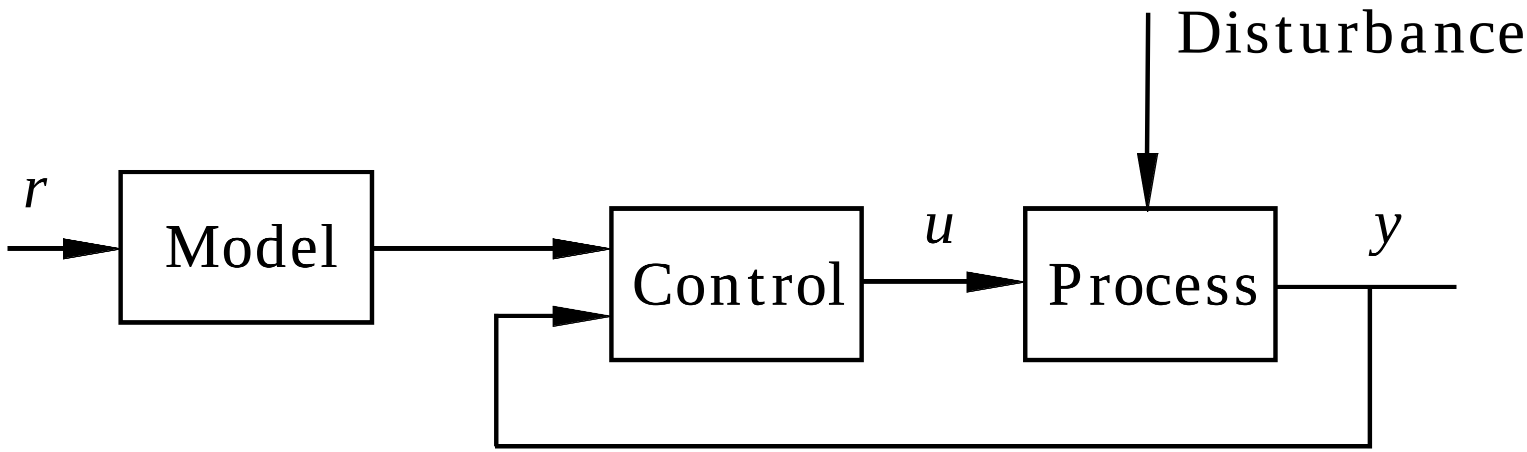
Use of **models** and **feedback**

Activities:

- Modeling
- Analysis and simulation
- Control design
- Implementation



Basic setting



Must handle two tasks:

- Follow reference signals, r
- Compensate for disturbances

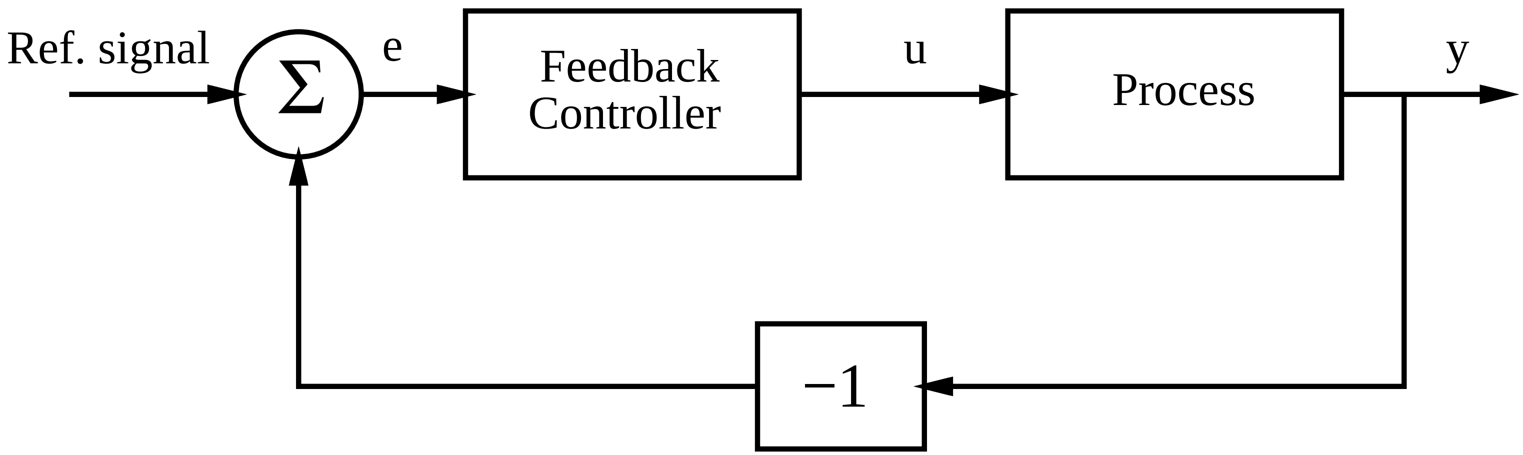
How to

- do several things with the control signal u

The feedback principle

A very powerful idea, that often leads to revolutionary changes in the way systems are designed.

The primary paradigm in automatic control.



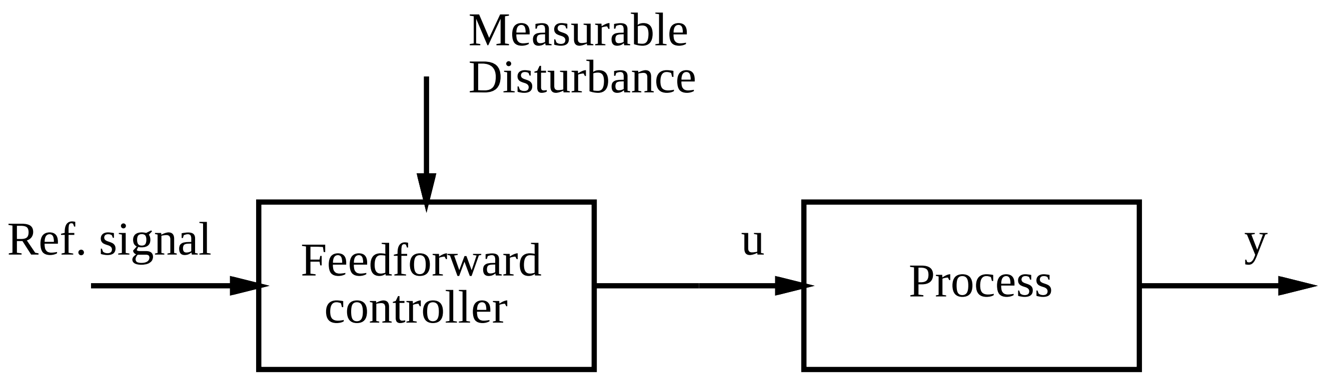
- Base corrective action on an error that has occurred
- Closed loop

Properties of feedback

+ Reduces influence of disturbances
+ Reduces effect of process variations
+ Does not require exact models

– Feeds sensor noise into the system
– May lead to instability, e.g.:
  - if the controller has too high gain
  - if the feedback loop contains too large time delays

The feedforward principle

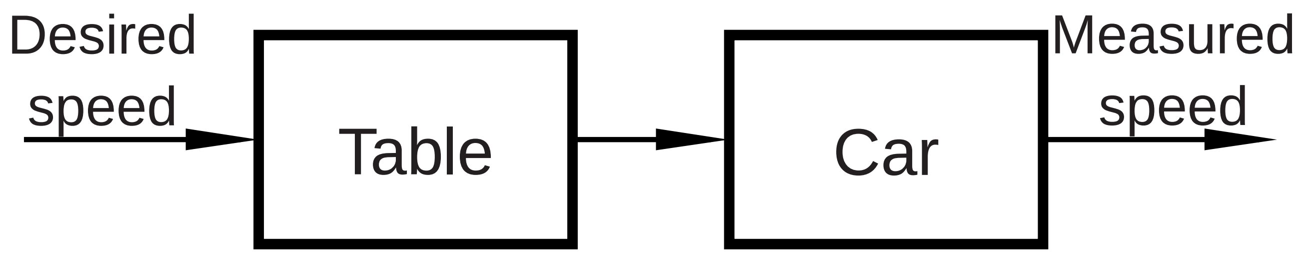


- Take corrective action before an error has occurred
- Measure the disturbance and compensate for it
- Use the fact that the reference signal is known and adjust the control signal to the reference signal
- Open loop

Properties of feedforward

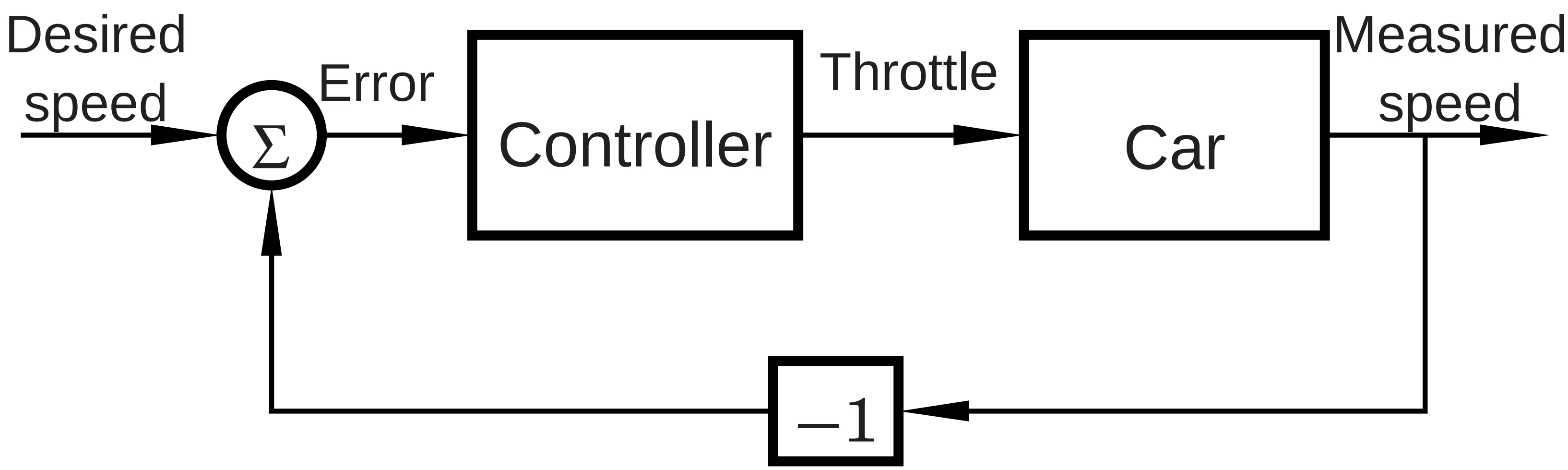
+ Reduces effect of disturbances that cannot be reduced by feedback
+ Allows faster set-point changes, without introducing control errors
– Requires good models
– Requires stable systems

Example: Cruise control using feedforward



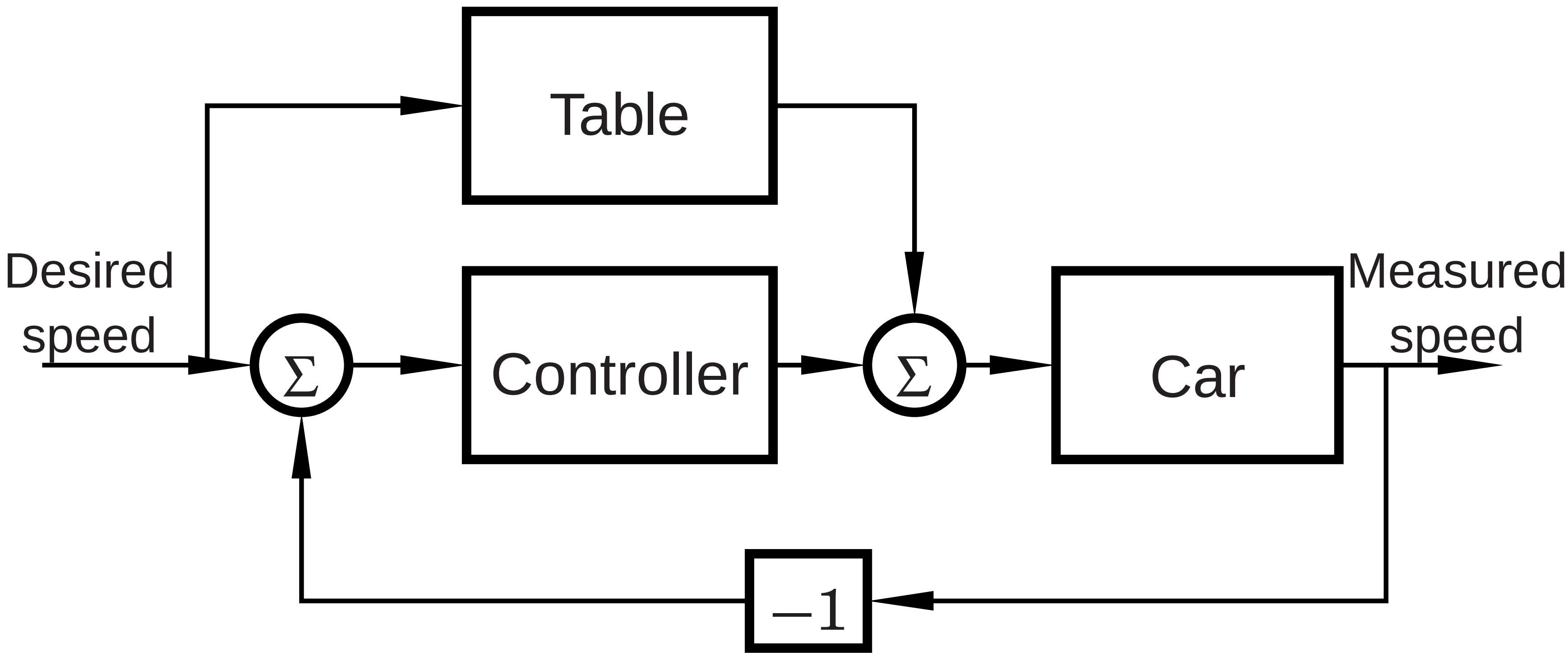
- Open loop
- Problems?

Example: Cruise control using feedback



- Closed loop
- Simple controller:
  - Error > 0: increase throttle
  - Error < 0: decrease throttle

Exempel: Cruise control using feedback and feedforward



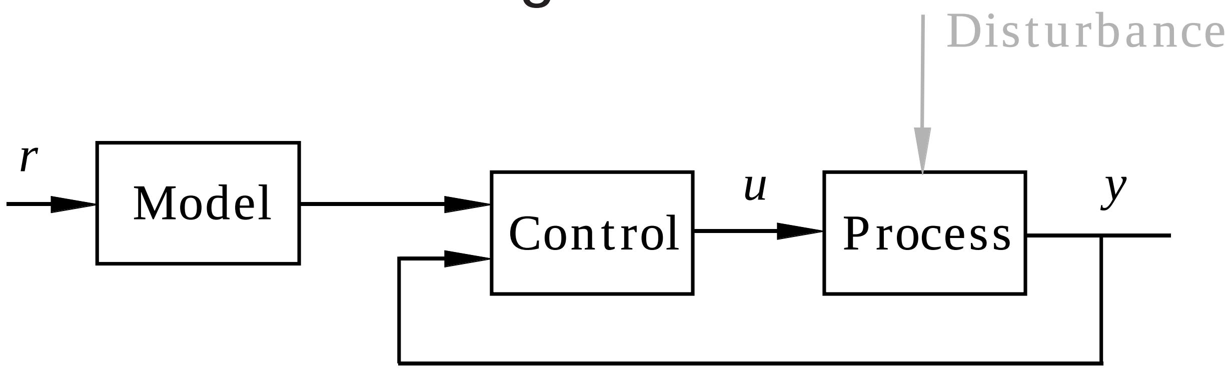
- Both proactive and reactive

Example: Segway

- Why is this process more difficult to control?
  - Unstable dynamics

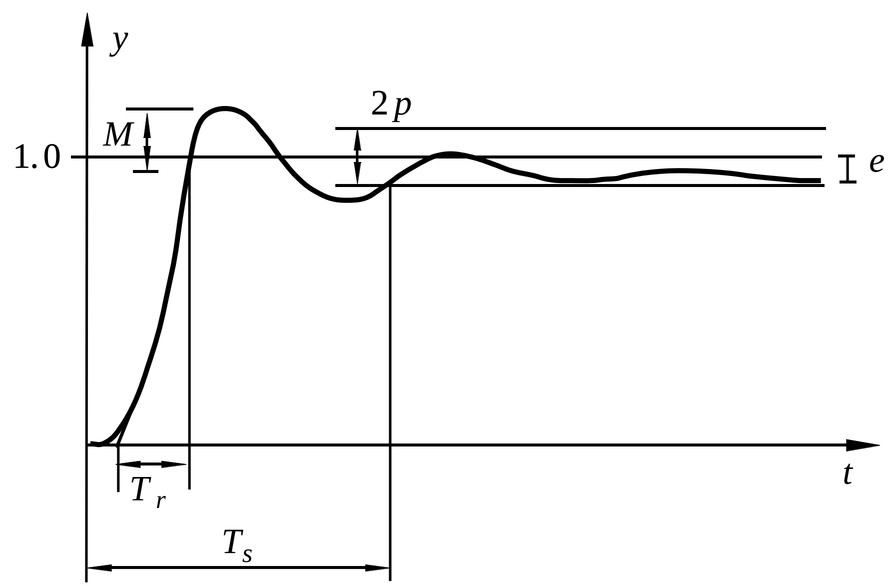
The servo problem

Focus on reference value changes:



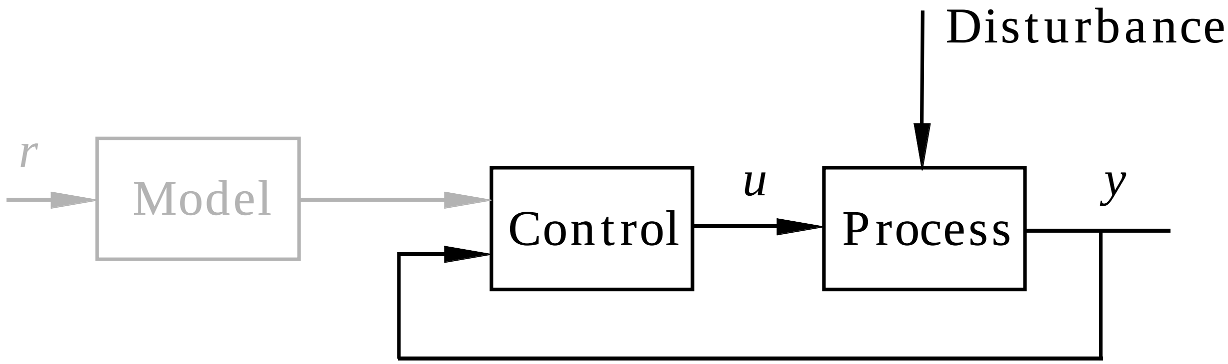
Typical design criteria:

- Rise time, T_r
- Overshoot, M
- Settling time, T_s
- Steady-state error, e_0
- . . .



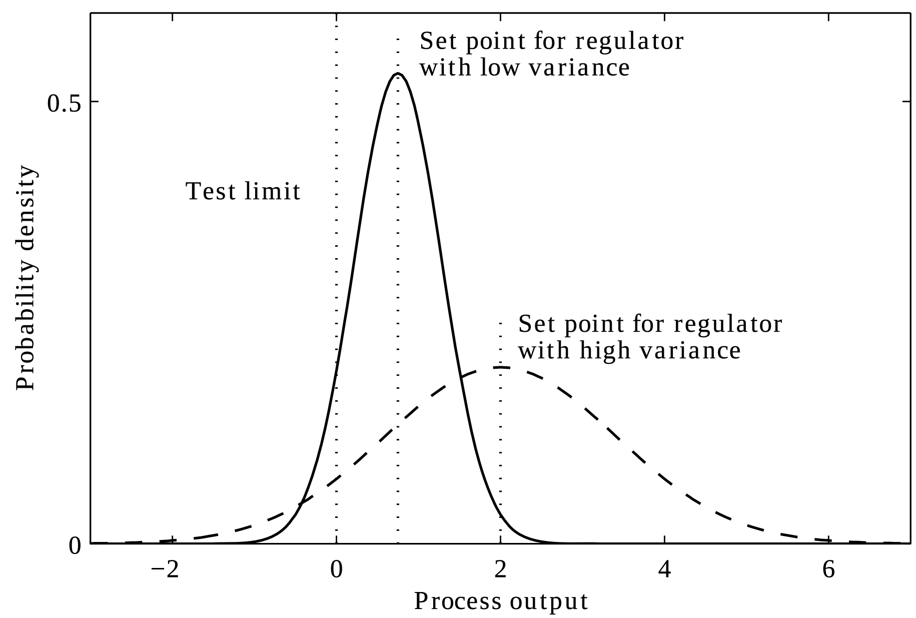
The regulator problem

Focus on process disturbances:



Typical design criteria:

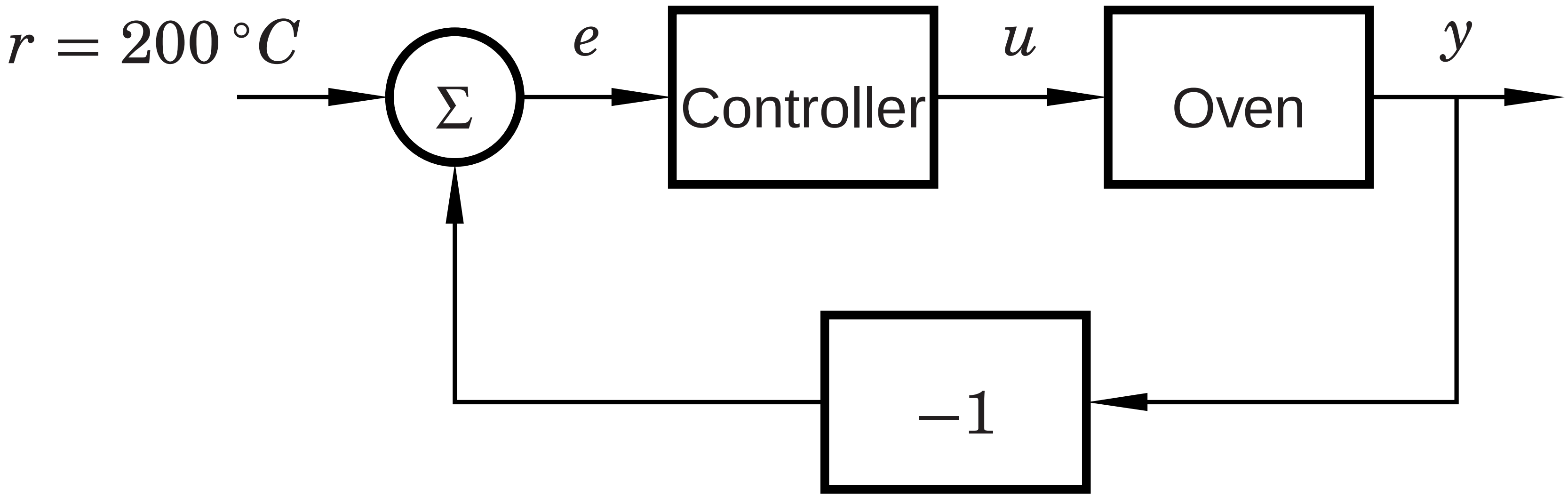
- Output variance
- Control signal variance



Session outline

- Feedback and feedforward
- **PID Control**
- State-space models
- Transfer function models
- Control design using pole placement
- State feedback and observers

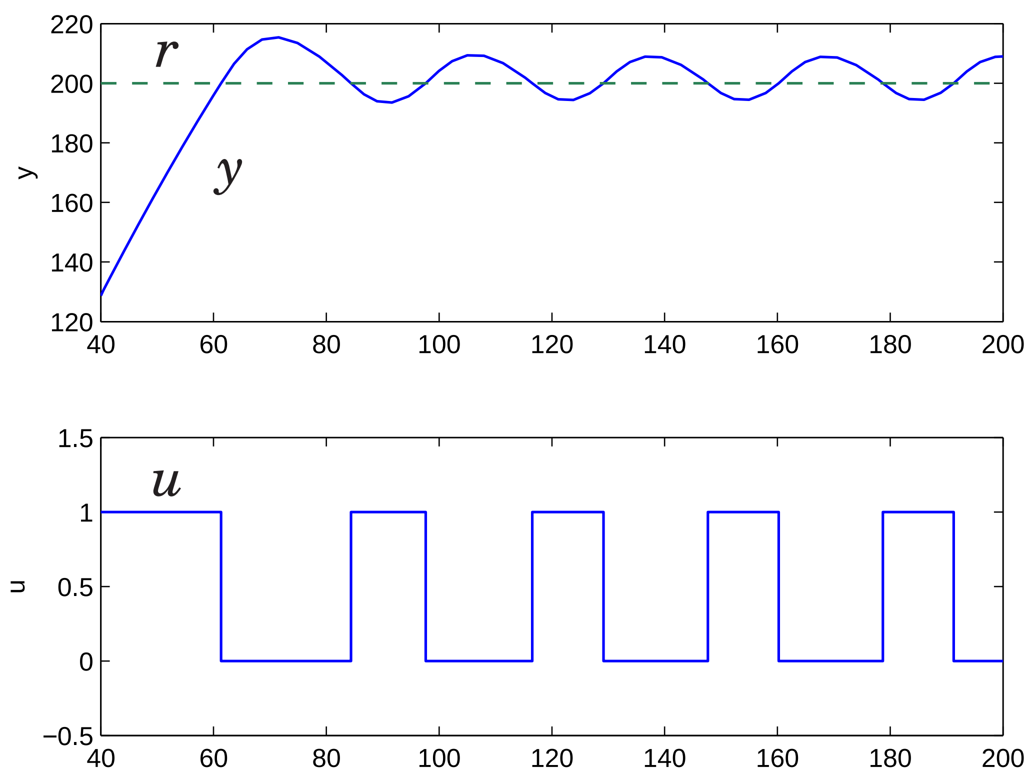
Example: Oven



- y – actual temperature
- r – desired temperature
- u – heating element power $(0 \le u \le 1)$

On/off control

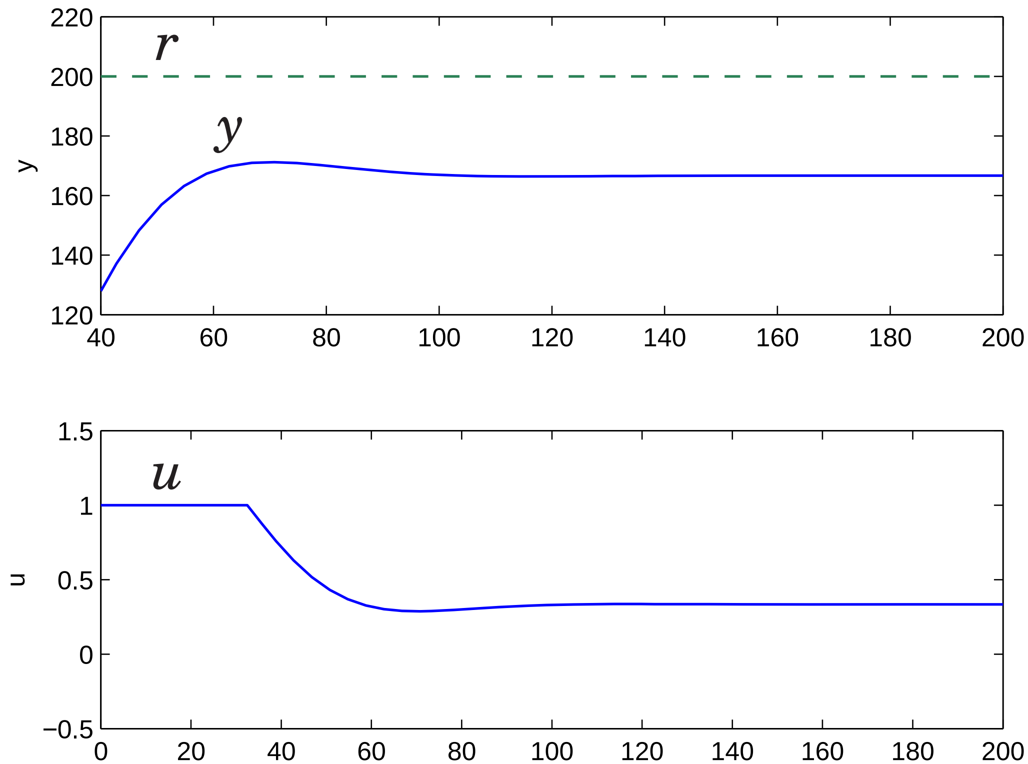
$$u(t) = \begin{cases} u_{\min}, & e(t) < 0 \\ u_{\max}, & e(t) > 0 \end{cases}$$



- Oscillations

Proportional control

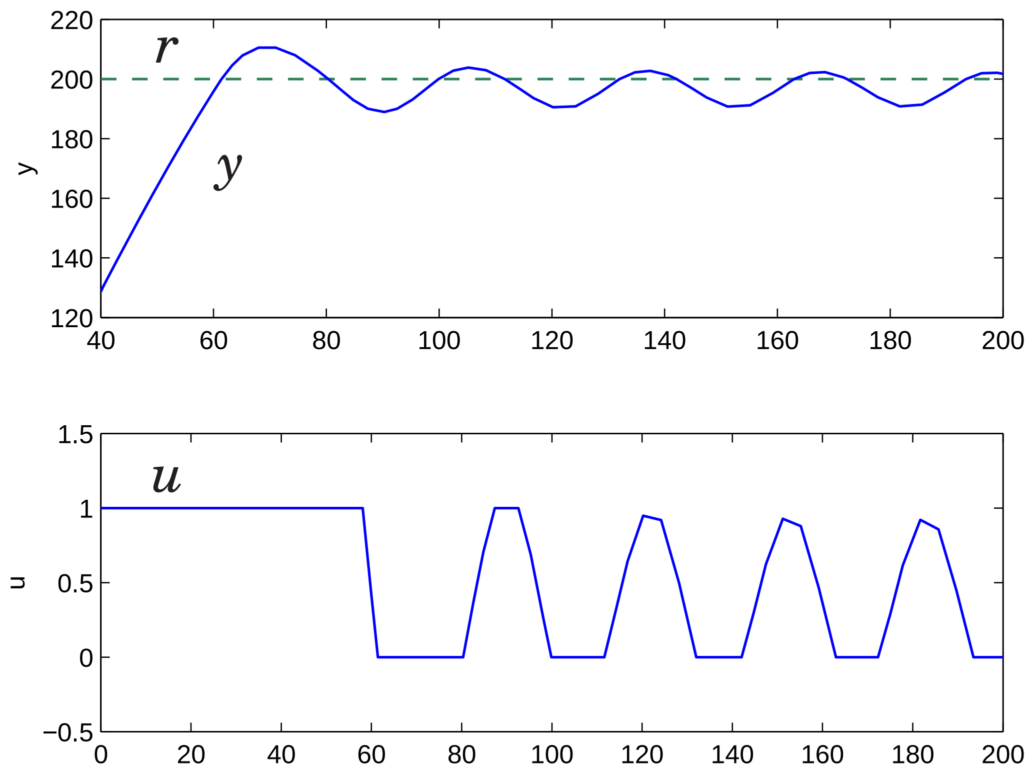
P-controller: $u(t) = Ke(t)$ (K – gain)



- Stationary error

Proportional control

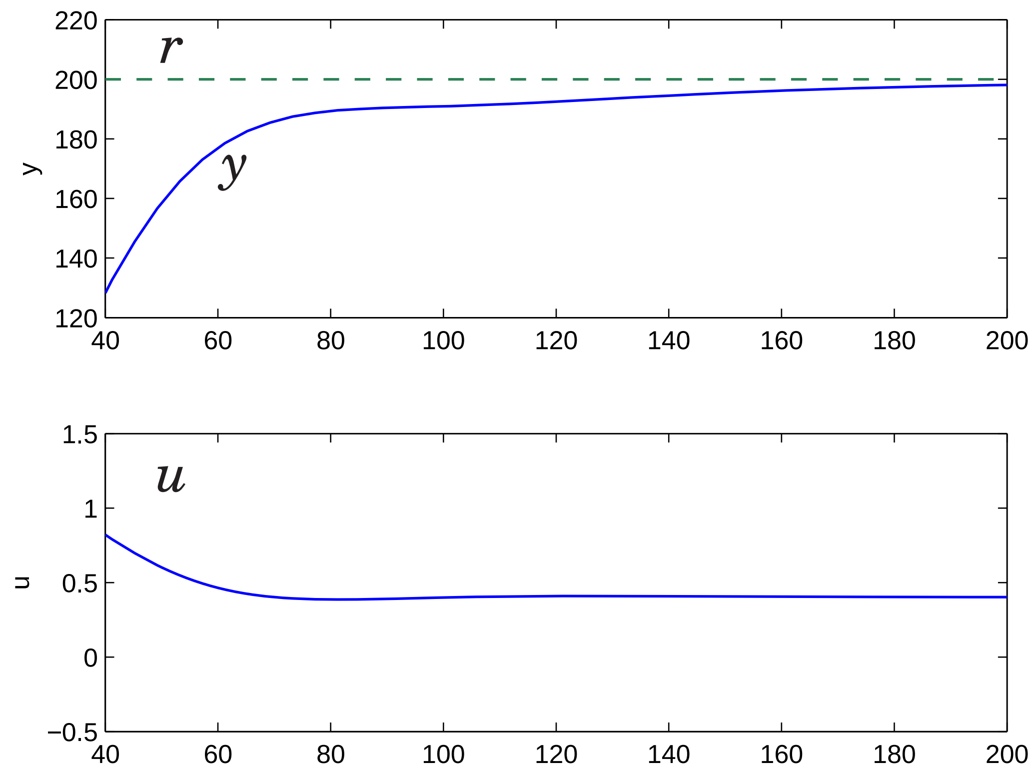
Increased gain K:



- Smaller stationary error
- Larger oscillations

Proportional–Integral control

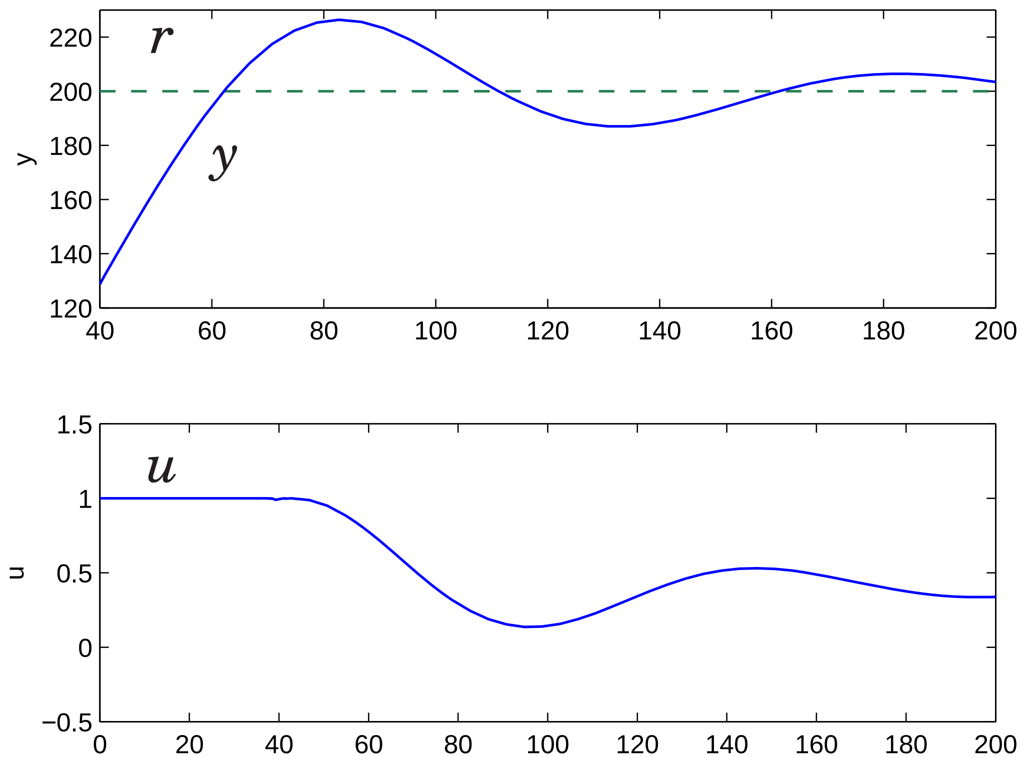
PI-controller: $u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(s)ds\right)$ (T_i – integral time)



- No stationary error

Proportional–Integral control

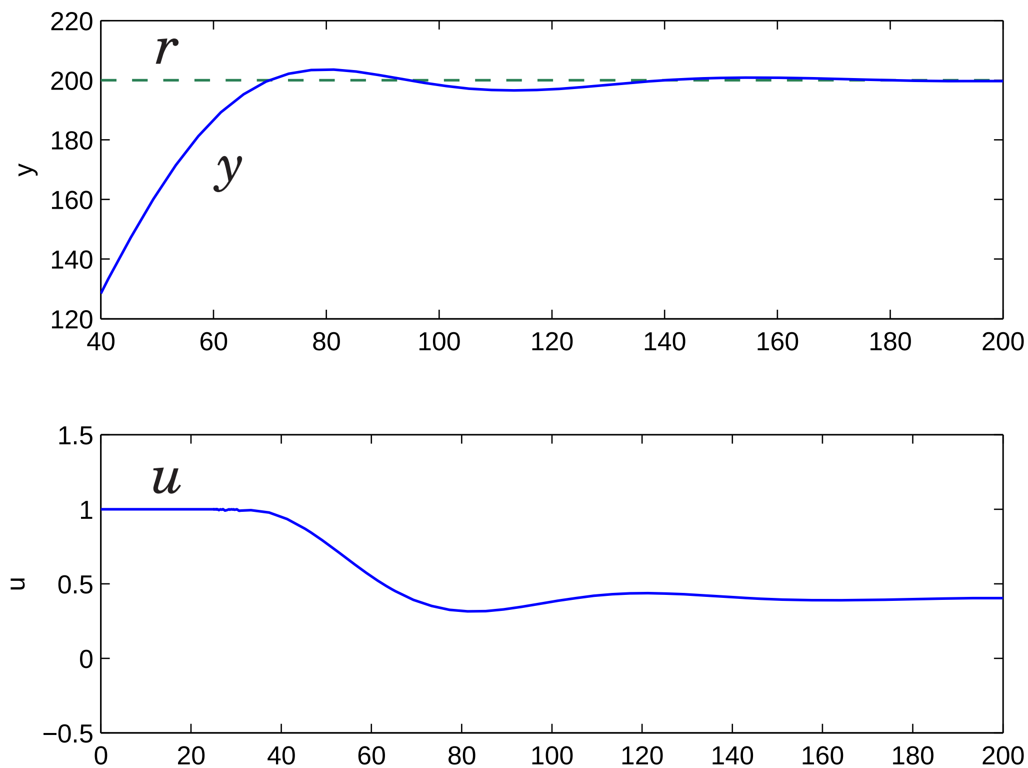
Smaller integral time T_i (larger integral action):



- Larger oscillations

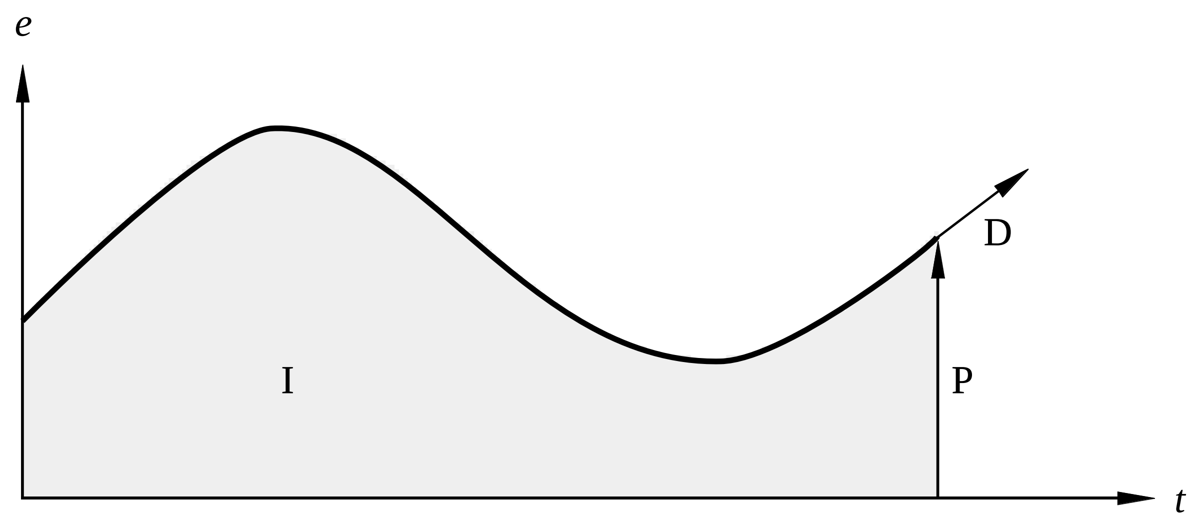
Proportional–Integral–Derivative control

PID-controller: $u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(s)ds + T_d\frac{de(t)}{dt}\right)$ (T_d – derivative time)



- The derivative part reduces oscillations

PID: Present, past, and future



- P-part: needed for fast response
- I-part: needed to remove stationary error
- D-part: may be needed to stabilize the process

Session outline

- Feedback and feedforward
- PID Control
- **State-space models**
- Transfer function models
- Control design using pole placement
- State feedback and observers

Dynamical systems

Static system:

$$y(t) = f\big(u(t)\big)$$

(The output at time t only depends on the input at time t.)

Dynamical system:

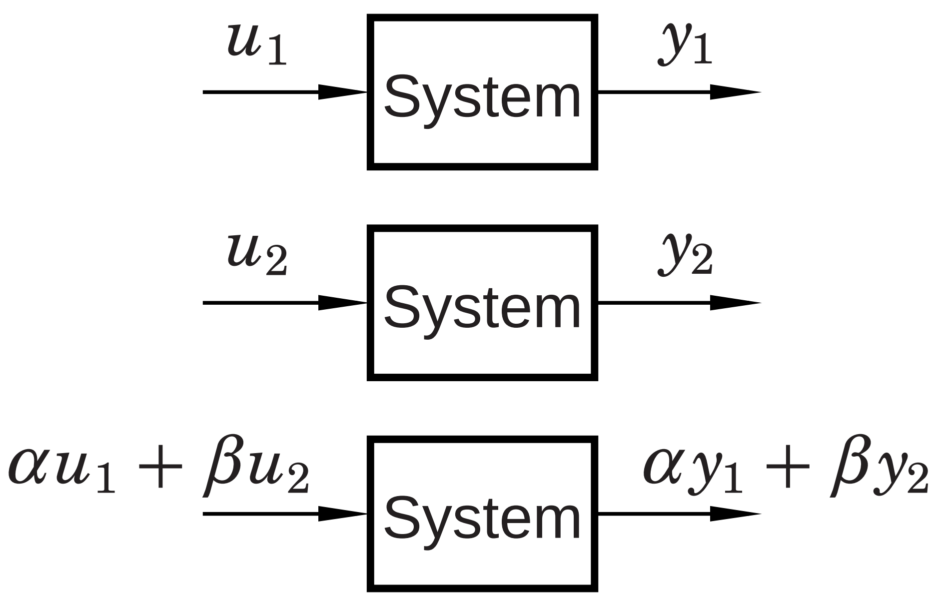
$$y(t) = f\big(x(0),\, u_{[0,\,t]}\big)$$

(The output at time t depends on the initial state $x(0)$ and the input from time 0 to t.)

Linear systems

We will mainly deal with linear, time-invariant (LTI) systems

For linear systems, the principle of superposition holds:



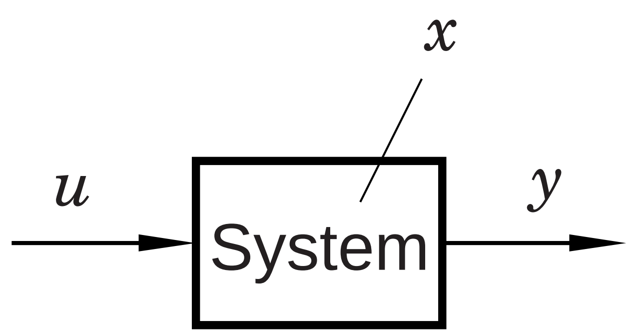
Nonlinear systems

- Almost all real systems are nonlinear
  - limited input and output signals
  - nonlinear process geometry
  - friction, turbulence, . . .
- Can be linearized around an operating point
- If there is feedback, a simple linear model is often enough
- But, always remember the limitations of the model!

Standard system forms

- State space form
  - A number of first-order differential equations
  - Describes what happens “inside” the system and how inputs and output are connected to this
  - Numerically superior
  - The heritage of mechanics
- Transfer function form
  - The transform of a higher-order linear differential equation
  - Describes the relationship between the input and the output
  - The system is a “black box”
  - Compact notation, convenient for hand calculations
  - The heritage of electrical engineering

State Space Models



Nonlinear state-space model:

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, \dots, x_n, u) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, \dots, x_n, u) \\
y &= g(x_1, \dots, x_n, u)
\end{aligned}
$$

Linear state-space model:

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + \dots + a_{1n}x_n + b_1 u \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \dots + a_{nn}x_n + b_n u \\
y &= c_1 x_1 + \dots + c_n x_n + du
\end{aligned}
$$

State Space Models

Introduce vectors and matrices for compact notation:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

n – system order

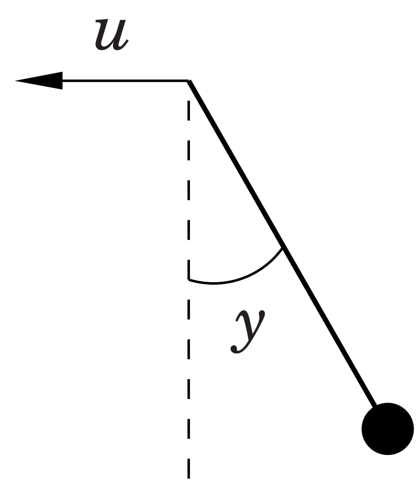
Nonlinear state-space model:

$$\frac{dx}{dt} = f(x, u)$$
$$y = g(x, u)$$

Linear state-space model:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Example: Pendulum



Nonlinear state-space model (x_1 = angle, x_2 = angular velocity):

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega_0^2 \sin x_1 + k\, u \cos x_1 \\
y &= x_1
\end{aligned}
$$

Linearization

A nonlinear system can be linearized around an equilibrium point, where it holds

$$\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad f(x^0, u^0) = 0$$

- Make first-order Taylor approximations of f and g around (x^0, u^0):

$$f(x,u) \approx \underbrace{f(x^0,u^0)}_{=0} + \frac{\partial f}{\partial x}\Big|_{(x^0,u^0)}(x - x^0) + \frac{\partial f}{\partial u}\Big|_{(x^0,u^0)}(u - u^0)$$

$$g(x,u) \approx \underbrace{g(x^0,u^0)}_{=y^0} + \frac{\partial g}{\partial x}\Big|_{(x^0,u^0)}(x - x^0) + \frac{\partial g}{\partial u}\Big|_{(x^0,u^0)}(u - u^0)$$

Linearization

- Introduce new variables $\Delta x = x - x^0$, $\Delta u = u - u^0$ och $\Delta y = y - y$
- The system can now be written as

$$\frac{d\Delta x}{dt} = \frac{dx}{dt} = f(x,u) \approx \left.\frac{\partial f}{\partial x}\right|_{(x^0,u^0)} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{(x^0,u^0)} \Delta u$$

$$\Delta y = g(x,u) - y^0 \quad \approx \left.\frac{\partial g}{\partial x}\right|_{(x^0,u^0)} \Delta x + \left.\frac{\partial g}{\partial u}\right|_{(x^0,u^0)} \Delta u$$

In matrix form:

$$\frac{d\Delta x}{dt} = A\Delta x + B\Delta u$$

$$\Delta y = C\Delta x + D\Delta u$$

Example – Pendulum

Linearize

$$\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1, x_2, u) \\
\dot{x}_2 &= -\omega_0^2 \sin x_1 + k\, u \cos x_1 && = f_2(x_1, x_2, u) \\
y &= x_1 && = g(x_1, x_2, u)
\end{aligned}$$

around the upper (unstable) equilibrium $x_1^0 = \pi,\ x_2^0 = 0,\ u^0 = 0$.

The linearized system is given by

$$\begin{aligned}
\frac{d\Delta x}{dt} &= A\Delta x + B\Delta u \\
\Delta y &= C\Delta x + D\Delta u
\end{aligned}$$

where $\Delta x = x - x^0$, $\Delta u = u - u^0$, $\Delta y = y - y^0$ and

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(x^0,u^0)} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 \cos x_1 - ku \sin x_1 & 0 \end{pmatrix}_{(x^0,u^0)}$$

$$= \begin{pmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}_{(x^0,u^0)} = \begin{pmatrix} 0 \\ k \cos x_1 \end{pmatrix}_{(x^0,u^0)} = \begin{pmatrix} 0 \\ -k \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}_{(x^0,u^0)} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$D = \frac{\partial g}{\partial u} = 0$$

Solving the system equation

The solution to the system equation

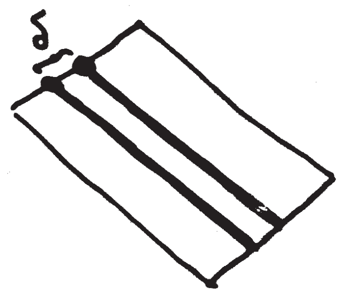
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

is given by

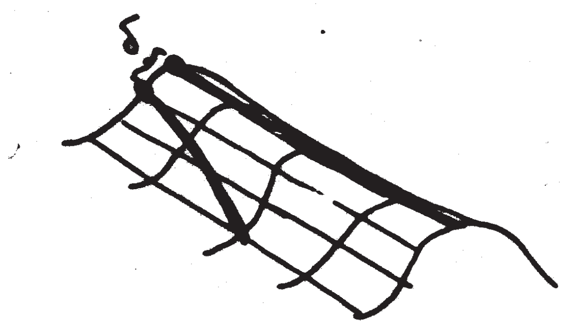
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

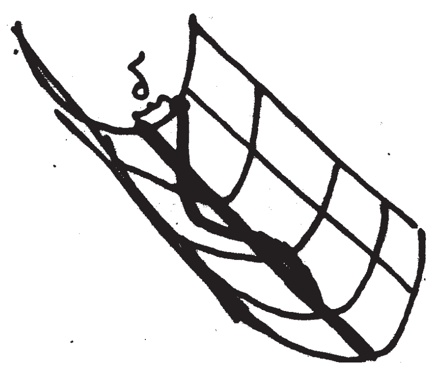
Stability concepts



Stable



Unstable



Asymptotically stable

Stability definitions

Assume

$$\dot{x} = Ax, \quad x(0) = x_0$$

The system is **stable** if $x(t)$ is limited for all x_0.

The system is **asymptotically stable** if $x(t) \rightarrow 0$ for all x_0.

The system is **unstable** if $x(t)$ is unlimited for some x_0.

Stability criteria

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \qquad \Rightarrow \qquad x(t) = x_0 e^{At}$$

The behavior of the solution depends on the eigenvalues of A

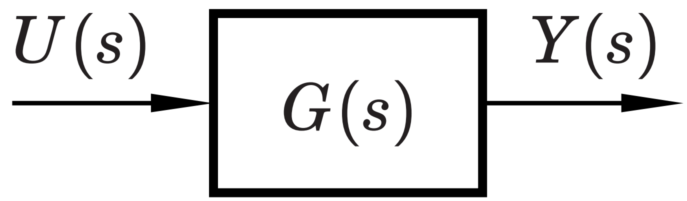
All eigenvalues have negative real part:	$\Leftrightarrow$	As. stab.
Some eigenvalue has positive real part:	$\Rightarrow$	Unstable
No eigenvalues with positive real part and no multiple eigenvalues on the imaginary axis:	$\Leftrightarrow$	Stable

Session outline

- Feedback and feedforward
- PID Control
- State-space models
- **Transfer function models**
- Control design using pole placement
- State feedback and observers

Transfer function form

Study the system in the (complex) frequency domain:



$U(s)$ – Laplace transform of $u(t)$

$Y(s)$ – Laplace transform of $y(t)$

$G(s)$ – transfer function

$$Y(s) = G(s)U(s)$$

(if the initial state is assumed to be zero)

Some operators/signals and their Laplace transforms

Definition: $$\mathcal{L}f = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Derivative: $$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s)$$

Integral: $$\mathcal{L}\left(\int f dt\right) = \frac{1}{s}F(s)$$

Dirac impulse: $$\mathcal{L}\delta = 1$$

Step function: $$\mathcal{L}\theta = \frac{1}{s}$$

Ramp function: $$\mathcal{L}(t\theta) = \frac{1}{s^2}$$

Exponential function: $$\mathcal{L}(e^{at}\theta) = \frac{1}{s-a}$$

From transfer function to state space form

$$\begin{cases} \dot{x} = Ax + Bu & x(0) = 0 \\ y = Cx + Du \end{cases}$$

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$Y(s) = \left[C(sI - A)^{-1}B + D\right] U(s)$$

$$G(s) = C(sI - A)^{-1}B + D = \frac{p(s)}{q(s)}$$

$q(s) = \det(sI - A)$ is called **characteristic polynomial**

Poles and zeros

Often,

$$G(s) = \frac{p(s)}{q(s)}$$

The roots of $p(s)$ are called zeros

The roots of $q(s)$ are called poles

Note that

$$\text{Poles of } G(s) \quad \Leftrightarrow \quad \text{Eigenvalues of } A$$

Calculating system responses

1. Find the transfer function $G(s)$ of the system
2. Find the Laplace transform $U(s)$ of the input $u(t)$
3. $Y(s) = G(s)U(s)$
4. Use inverse Laplace transform to find $y(t)$

Calculating system responses

Example:

Compute the step response of $G(s) = \frac{1}{s+1}$

Input: $U(s) = \mathcal{L}\{\theta(t)\} = \frac{1}{s}$

Output: $Y(s) = G(s)U(s) = \frac{1}{s(s+1)}$

Output in the time domain:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$$

Step response of first-order systems

$$G(s) = \frac{1}{s+a} \quad \Rightarrow \quad \text{step response} \quad y(t) = \frac{1}{a}(1 - e^{-at})$$

$$G(s) = \frac{1}{s+a} = \frac{T}{1+sT}$$

Time constant: $T = \dfrac{1}{a}$

Static gain: $G(0) = 1/a$

Step response of second-order systems

Real poles:

$$G(s) = \frac{1}{(s+a)(s+b)} \quad \Rightarrow \quad \text{step response:} \quad y(t) = \frac{1}{ab}\,\frac{ae^{-bt} - be^{-at}}{ab(b-a)}$$

Complex poles:

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} \quad \Rightarrow$$

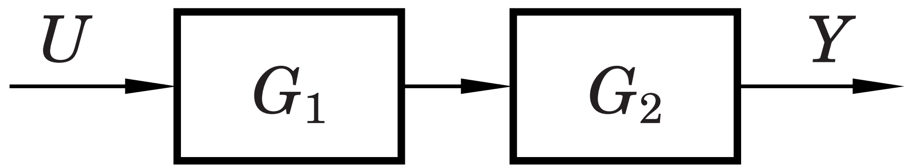
step response: $y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega\sqrt{1-\zeta^2}t + \phi)$

$\phi = \arccos\zeta$

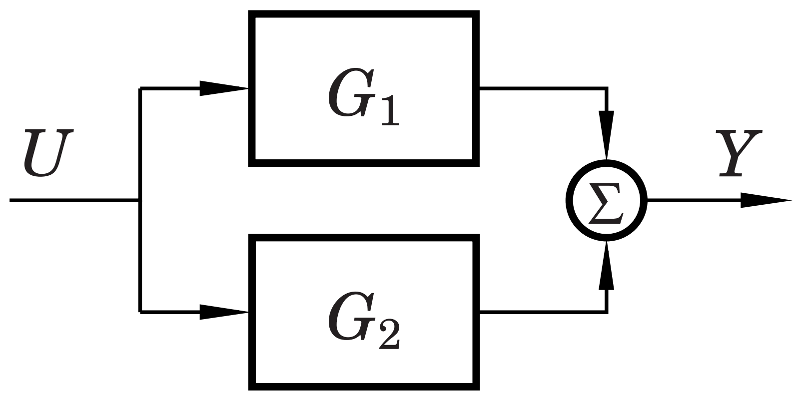
$\omega =$ undamped frequency $(\omega > 0)$

$\zeta =$ relative damping $(0 < \zeta < 1)$

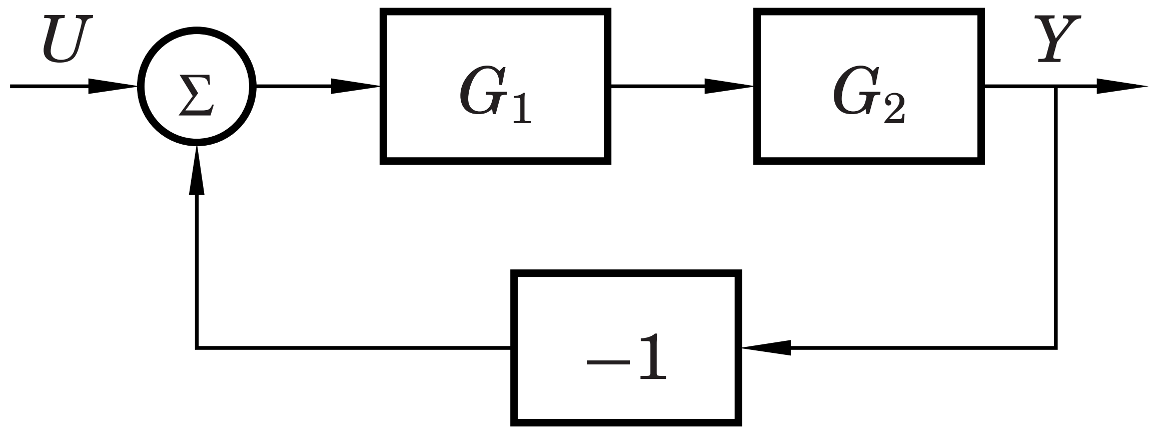
Block diagrams



$$Y = G_2 G_1 \, U$$

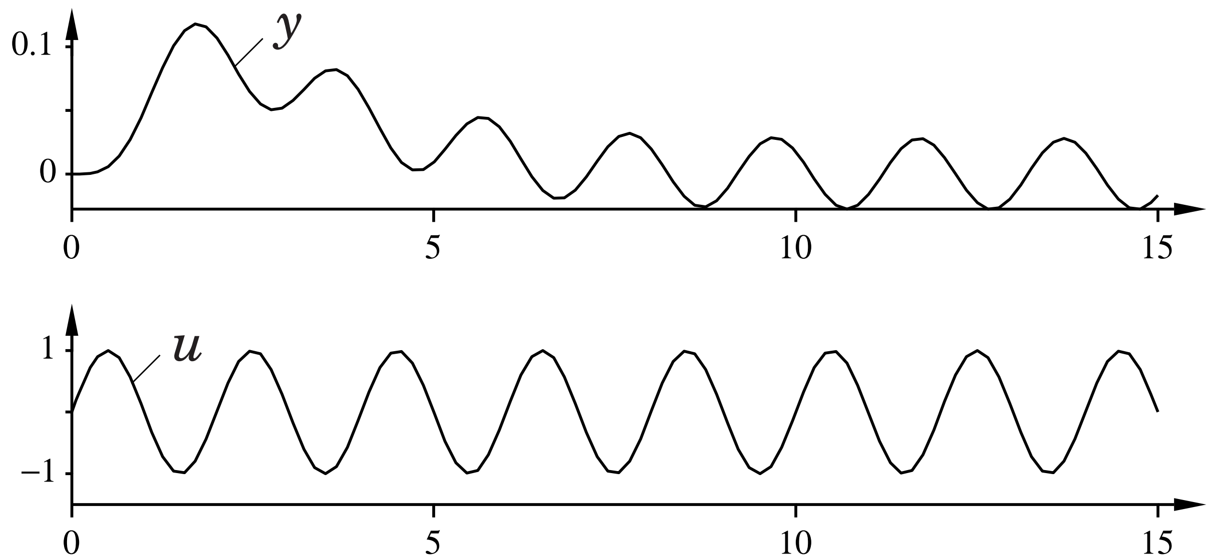


$$Y = \left(G_1 + G_2\right) U$$



$$
\begin{aligned}
Y &= G_2 G_1 (U - Y) \\
Y(1 + G_2 G_1) &= G_2 G_1 U \\
Y &= \frac{G_2 G_1}{1 + G_2 G_1} U
\end{aligned}
$$

Frequency response



Given a stable system $G(s)$, the input $u(t) = \sin \omega t$ will, after a transient, give the output

$$y(t) = |G(i\omega)| \sin \left(\omega t + \arg G(i\omega) \right)$$

The steady-state output is also sinusoidal

Bode diagram

Draw

- $|G(i\omega)|$ as a function of ω (in log-log scale)
  - Amplitude/magnitude/gain diagram
- $\arg G(i\omega)$ as a function of ω (in log-lin scale)
  - Phase/angle diagram

Example: low-pass filter

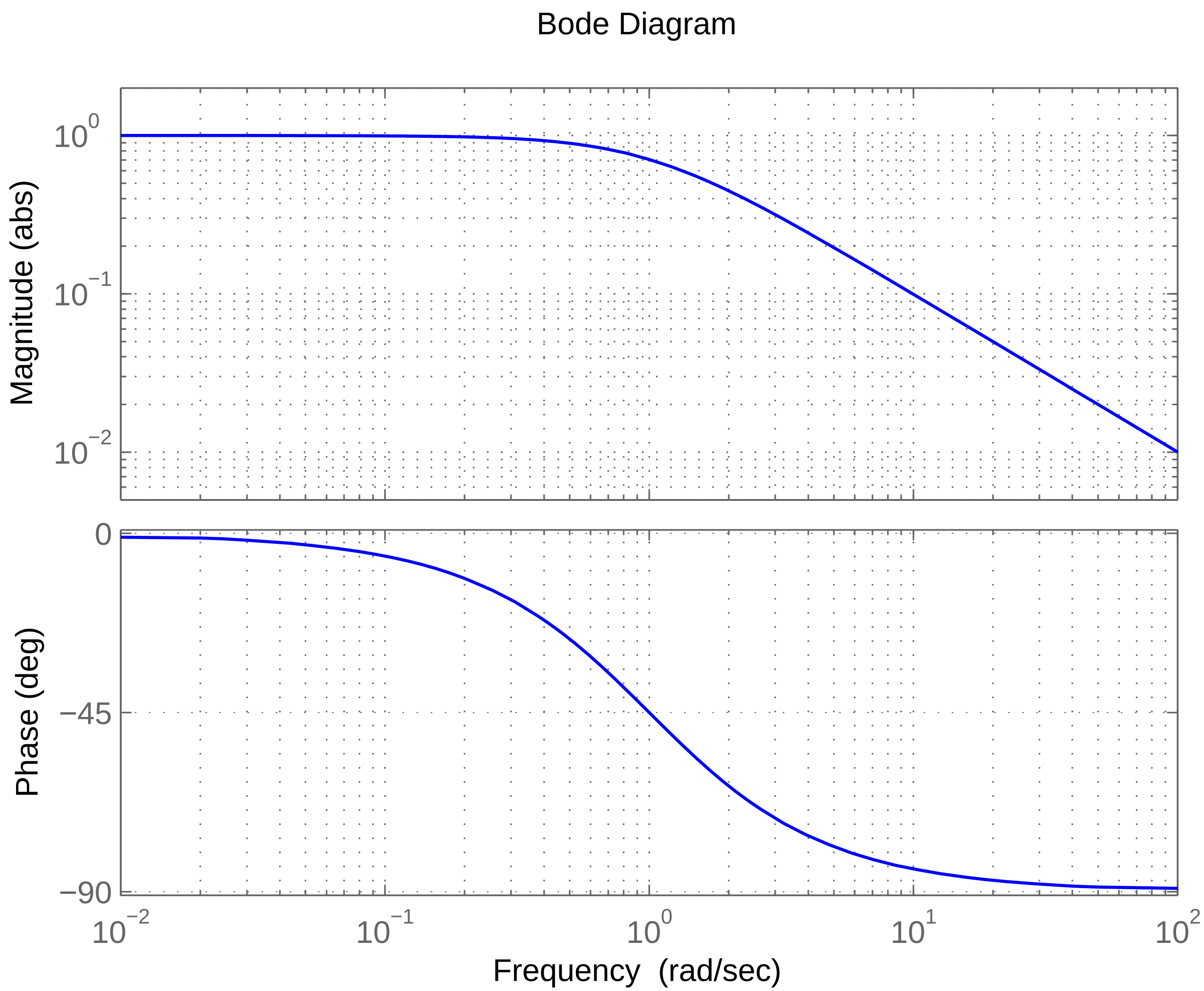
$$\frac{dy(t)}{dt} + y(t) = u(t) \quad \Leftrightarrow \quad G(s) = \frac{1}{s+1}$$

$$G(i\omega) = \frac{1}{i\omega + 1}$$

$$|G(i\omega)| = \frac{1}{\sqrt{\omega^2 + 1}}$$

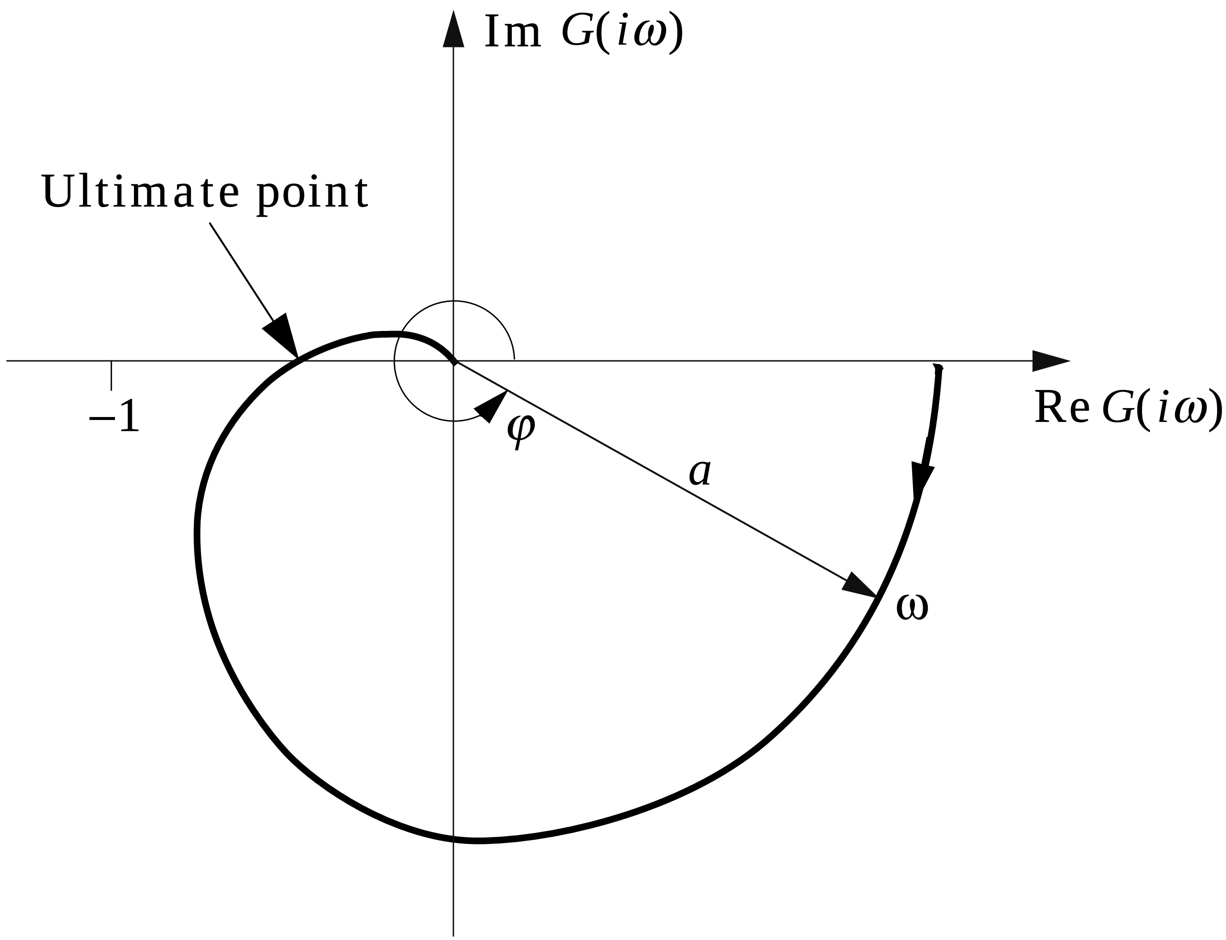
$$\arg G(i\omega) = -\arctan\omega$$

Example: low-pass filter

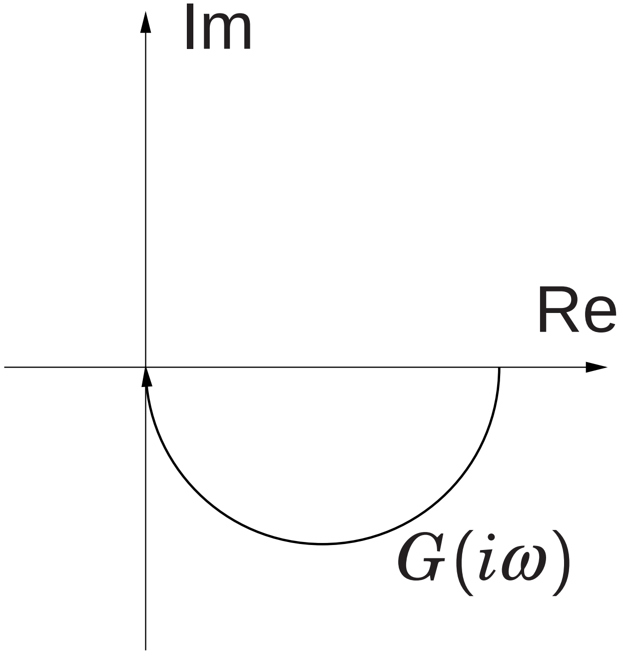


Nyquist Diagram

Draw $G(i\omega)$ in a polar diagram when ω goes from 0 to ∞



Example of Nyquist Diagram



$$G(s) = \frac{1}{s+1}$$

$$G(i\omega) = \frac{1}{i\omega + 1} = \frac{1 - i\omega}{\omega^2 + 1}$$

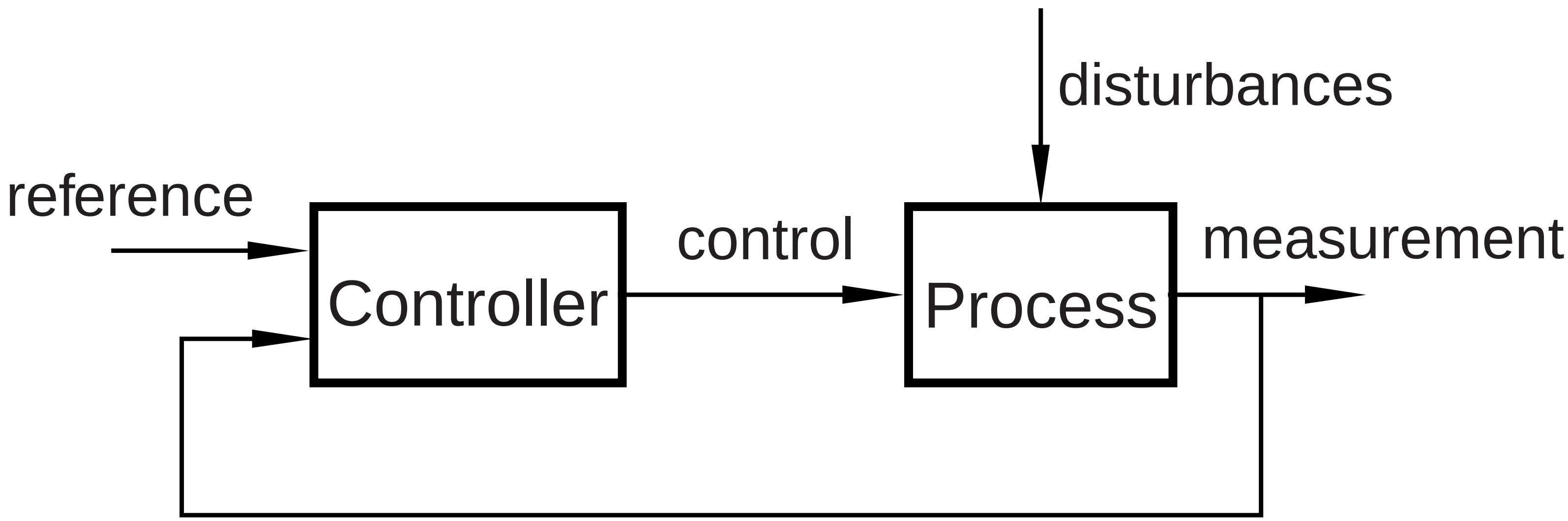
Small ω : $G(i\omega) \approx 1$

Large ω : $G(i\omega) \approx \frac{1}{\omega^2} - i\frac{1}{\omega}$

Session outline

- Feedback and feedforward
- PID Control
- State-space models
- Transfer function models
- **Control design using pole placement**
- State feedback and observers

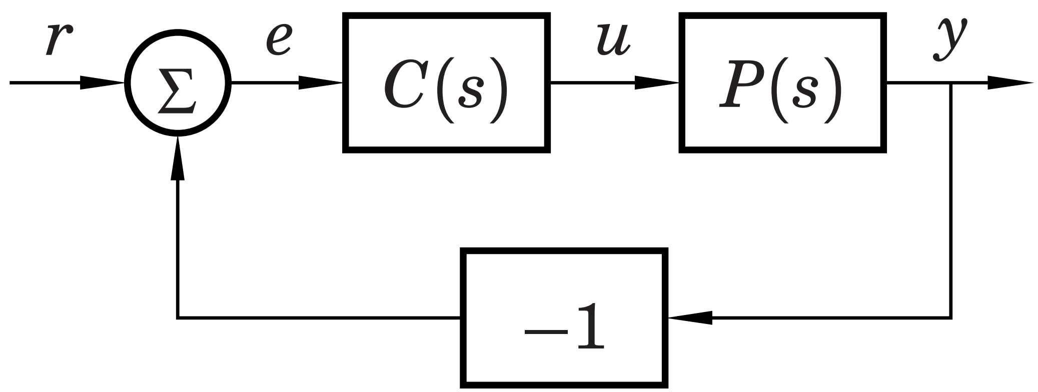
Closed-loop control



Primary goals of the controller:

- Follow the reference
- Reject disturbances

Analysis of the standard feedback loop



- $C(s)$: controller
- $P(s)$: process

Closed-loop transfer function (from r to y):

$$Y = \frac{PC}{1+PC}R$$

Control design: Choose C to get the desired behavior!

Example – cruise control

Assume that the relationship between the throttle and the speed is given by

$$\frac{dy}{dt} = -0.2y + 5u \;\Leftrightarrow\; P(s) = \frac{5}{s+0.2}$$

First try to regulate the speed with a P-controller:

$$u(t) = Ke(t)$$

where $e(t) = r(t) - y(t)$

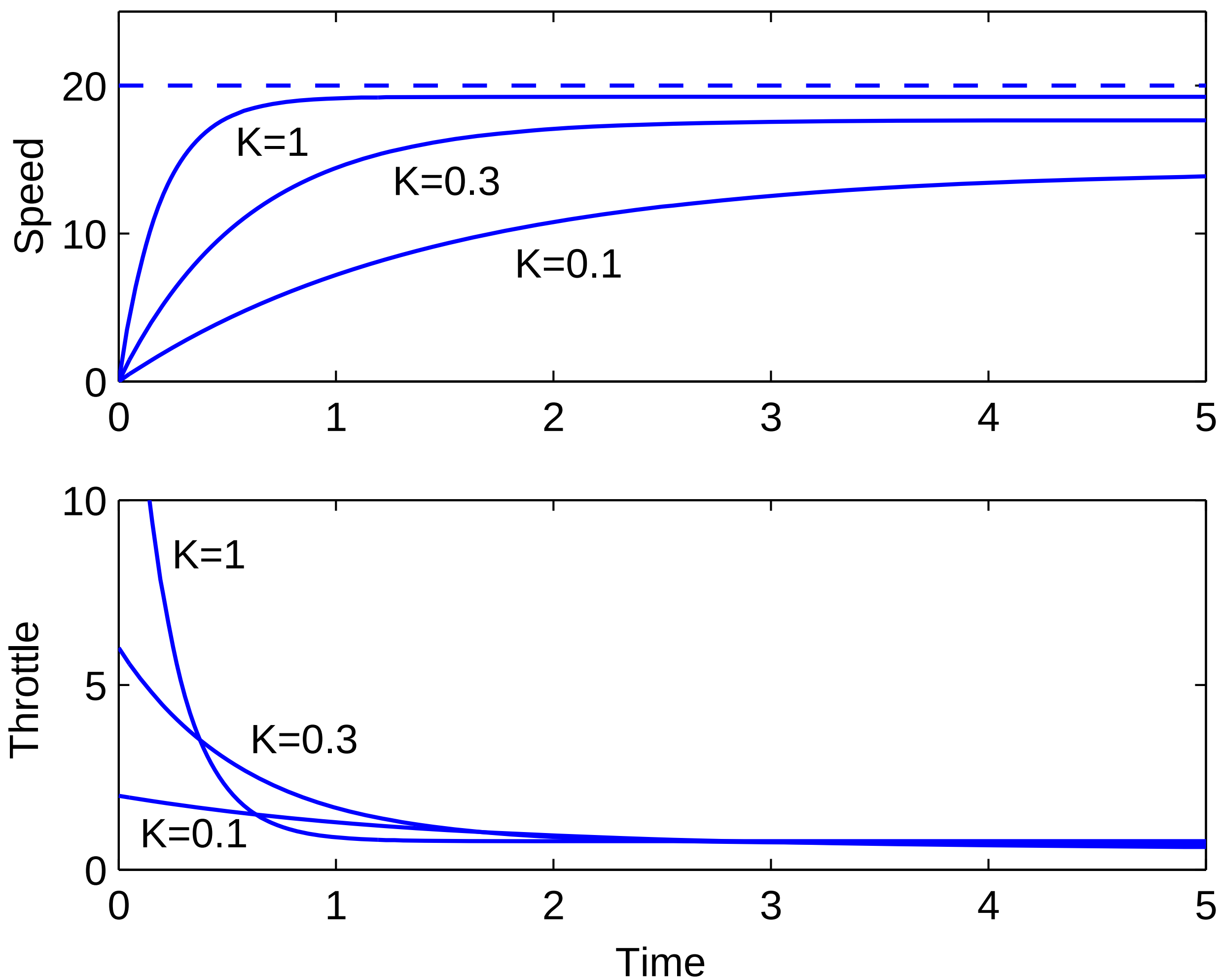
The closed-loop transfer function is given by

$$\frac{PC}{1+PC} = \frac{\frac{5}{s+0.2} \cdot K}{1 + \frac{5}{s+0.2} \cdot K} = \frac{5K}{s + 0.2 + 5K}$$

The gain K affects

- the pole of the closed-loop system
- the static gain of the closed-loop system

Simulation of the control system with different values of K:



- Stationary error

Now try a PI-controller:

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau\right)$$

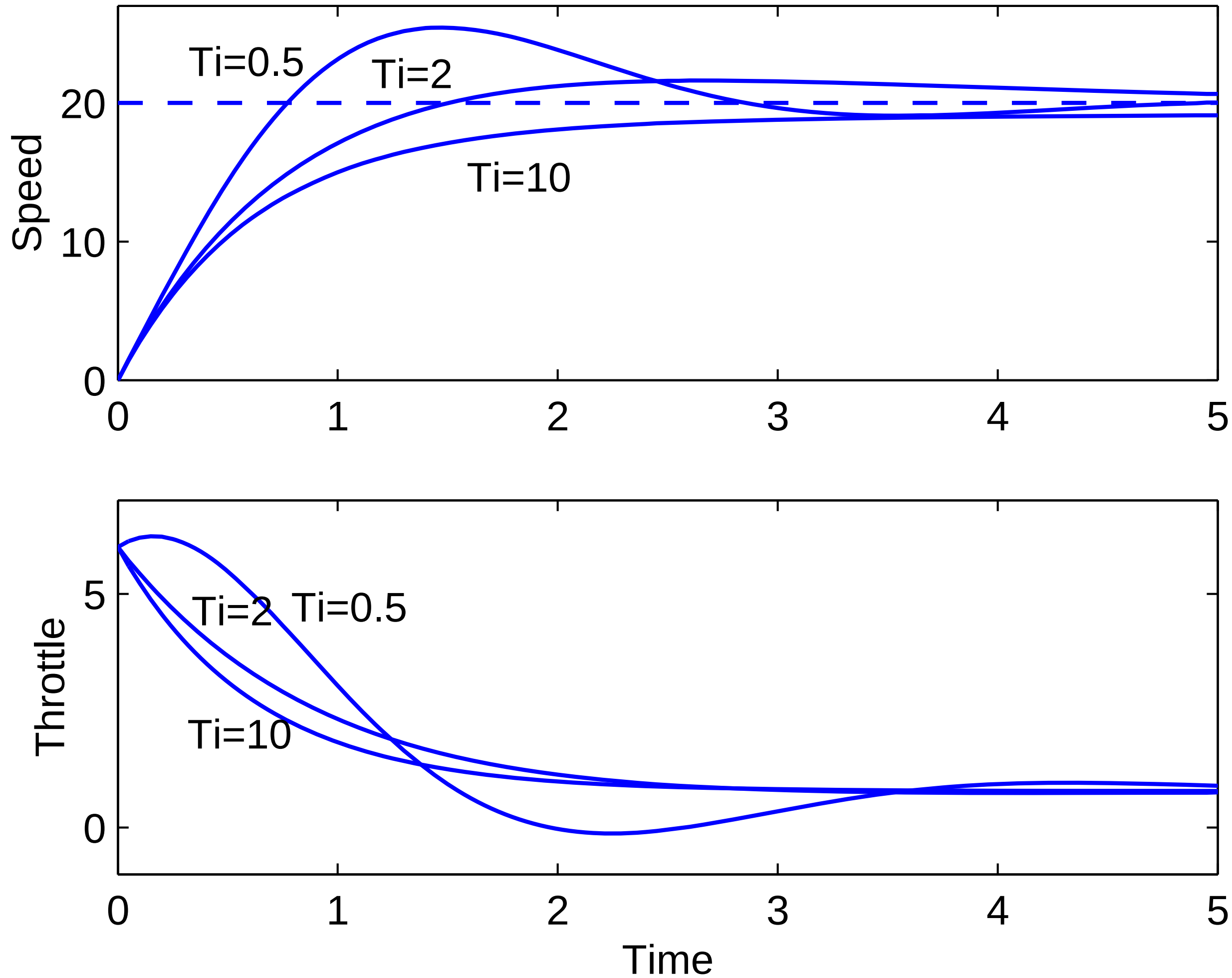
$$U(s) = \underbrace{K\left(1 + \frac{1}{sT_i}\right)}_{C(s)} E(s)$$

The closed-loop transfer function is given by

$$\frac{PC}{1+PC} = \frac{\frac{5}{s+0.2} \cdot K\left(1+\frac{1}{sT_i}\right)}{1+\frac{5}{s+0.2} \cdot K\left(1+\frac{1}{sT_i}\right)} = \frac{5K\left(s+\frac{1}{T_i}\right)}{s^2+(5K+0.2)s+\frac{5K}{T_i}}$$

- The poles of the closed-loop system depend on K and T_i
- The static gain of the closed-loop system is always 1

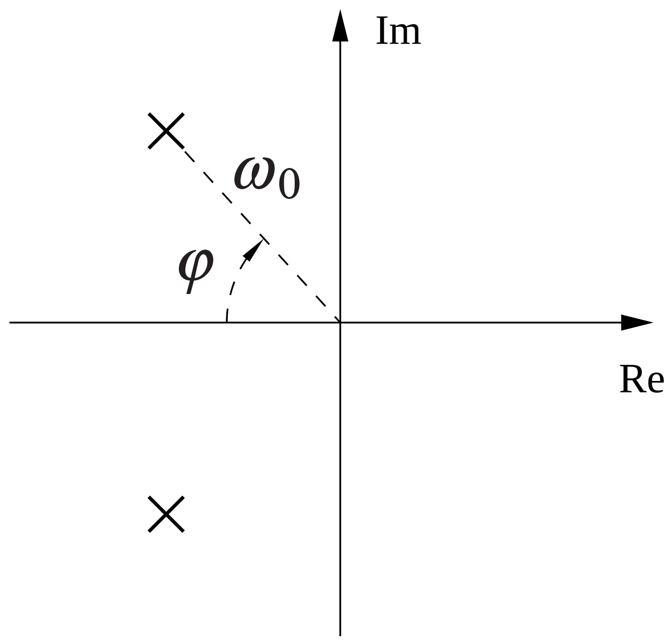
Simulation of the control system with $r = 20$, $K = 0.3$ and different values of T_i:



- No stationary error

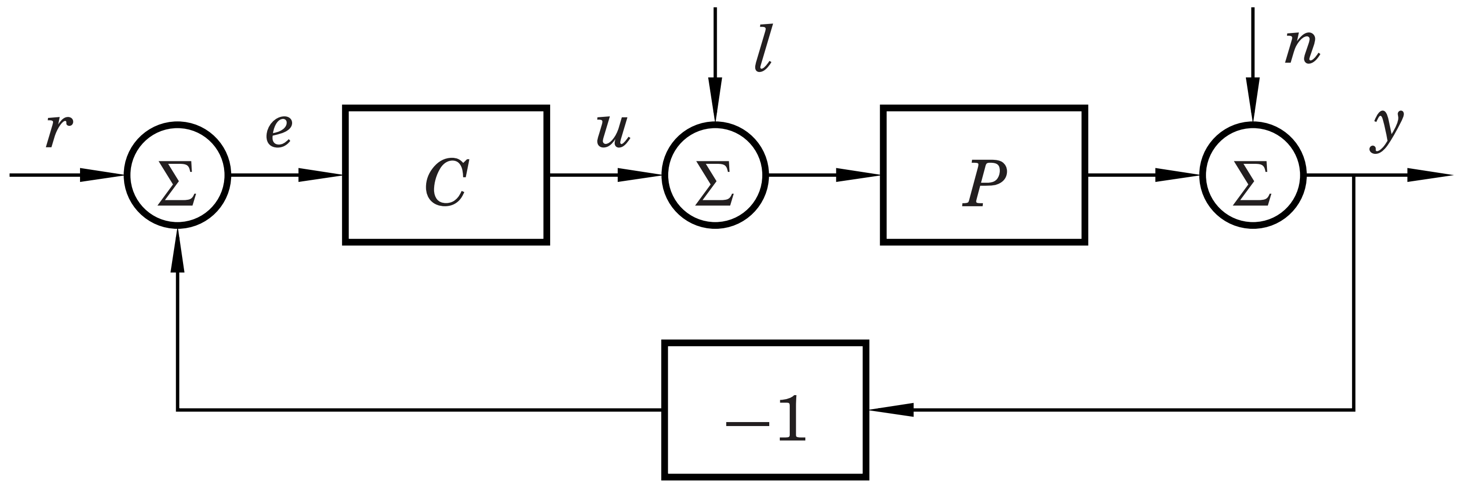
Where to place the poles?

Pole placement according to the characteristic polynomial $q(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$:



- Larger $\omega_0 \Rightarrow$ faster system response
- Smaller $\varphi \Rightarrow$ better damping (relative damping $\zeta = \cos\varphi$). (Common choice: $\zeta = \cos 45° = 0.7$)

Analysis of the standard loop with disturbances



- l: load disturbance
- n: noise

Influence of disturbances

From the block diagram the following relationships can be derived:

$$Y = \frac{PC}{1+PC}R + \frac{P}{1+PC}L + \frac{1}{1+PC}N$$

$$U = \frac{C}{1+PC}R - \frac{PC}{1+PC}L - \frac{C}{1+PC}N$$

$$E = \frac{1}{1+PC}R - \frac{P}{1+PC}L - \frac{1}{1+PC}N$$

Since the system is linear, we can analyze the influence of reference values, load disturbances, and measurement noise separately

Design trade-offs

Ideally, one would like to have

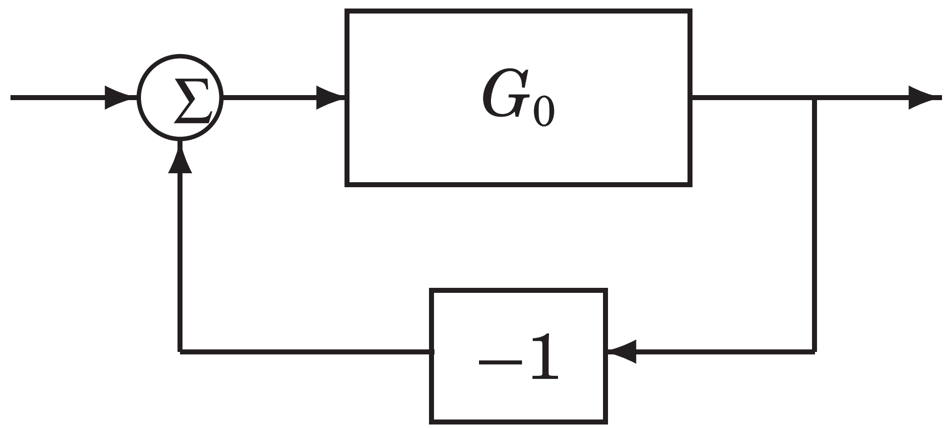
- perfect reference tracking, $\frac{PC}{1+PC} = 1$
- no influence of load disturbances, $\frac{P}{1+PC} = 0$
- no influence of measurement noise, $\frac{C}{1+PC} = 0$
- $\ldots$

Impossible to fulfill

Typical design compromise:

- $C(s)$ high gain at low frequencies
- $C(s)$ low gain at high frequencies

Stability under Feedback



The closed loop system is asymptotically stable if and only if all the zeros of

$$1 + G_0(s)$$

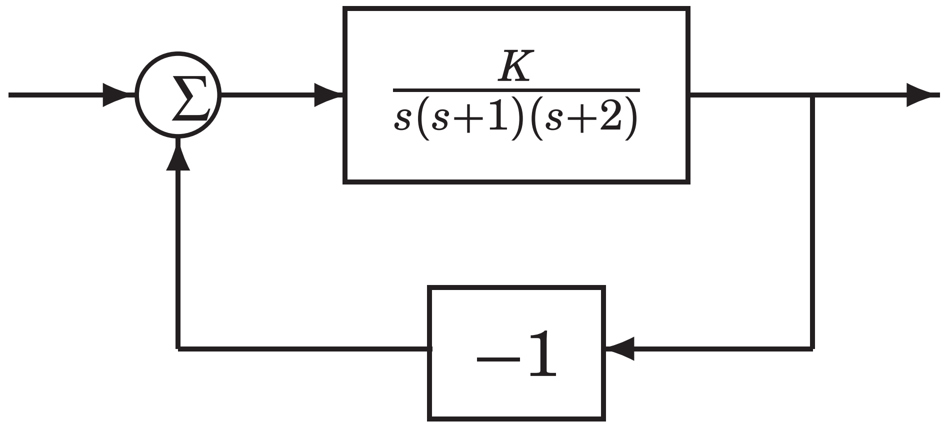
lies in the left half plane.

The Nyquist Criterion

If $G_0(s)$ is stable then the closed loop system $[1 + G_0(s)]^{-1}$ is stable if and only if the the Nyquist curve$G(i\omega)$ does not encircle -1.

$G_0(s) = G_P(s)G_R(s)$, i.e. modify G_P such that the Nyquist curve does not encircle -1

Example

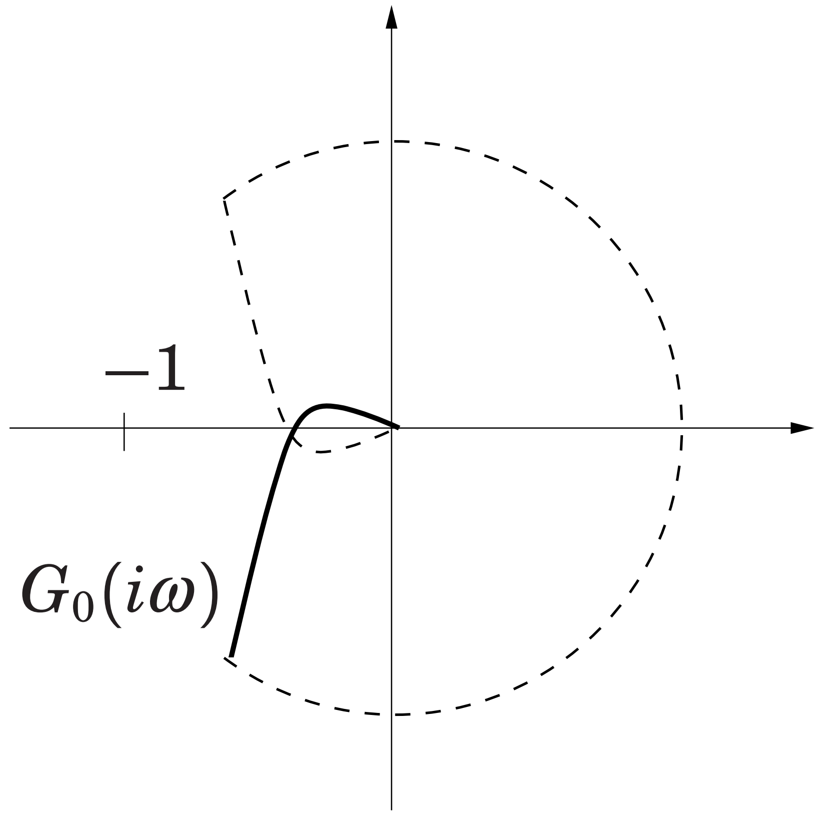


$$G_0(i\omega) = \frac{K}{i\omega(1+i\omega)(2+i\omega)}$$

$$= \frac{-Ki(1-i\omega)(2-i\omega)}{\omega(1+\omega^2)(4+\omega^2)} = \frac{-Ki(2-\omega^2-3i\omega)}{\omega(1+\omega^2)(4+\omega^2)}$$

$$= \frac{-3K}{(1+\omega^2)(4+\omega^2)} + i\frac{K(\omega^2-2)}{\omega(1+\omega^2)(4+\omega^2)}$$

Stability for the closed loop system



$$G_0(i\sqrt{2}) = -\frac{3K}{3 \cdot 6} = -\frac{K}{6}$$

Stable if $K < 6$.

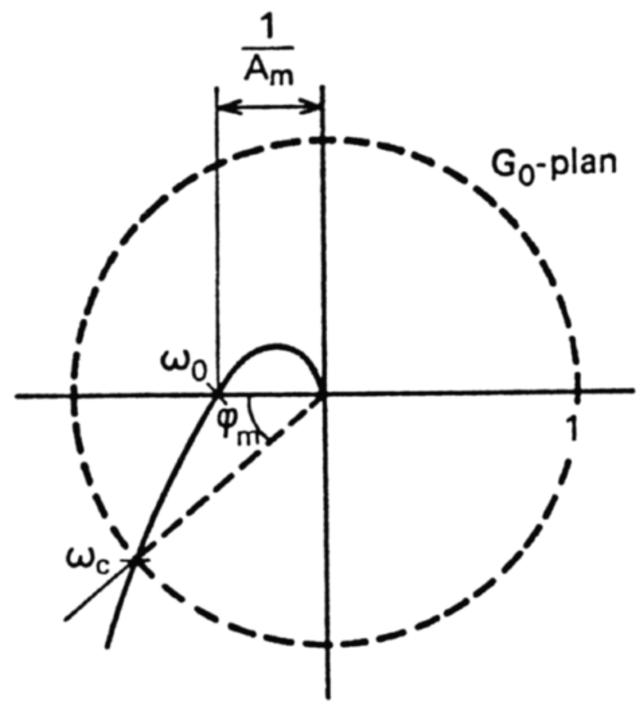
Amplitude and phase margins

Amplitude margin A_m

$$\arg G(i\omega_0) = -180^\circ, \quad |G(i\omega_0)| = \frac{1}{A_m}$$

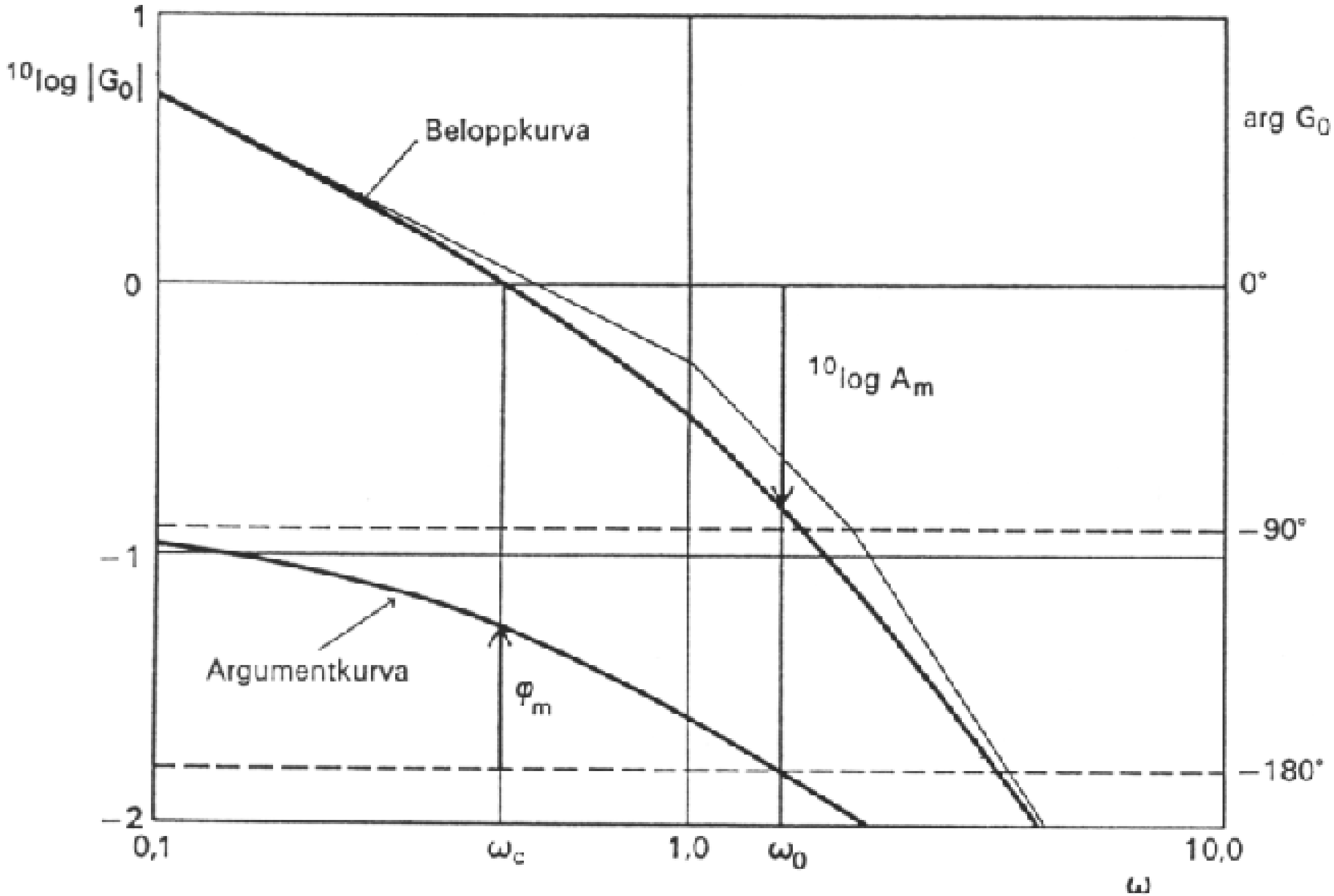
Phase margin ϕ_m

$$|G(i\omega_c)| = 1, \quad \arg G(i\omega_c) = \phi_m - 180^\circ$$



(Rules of Thumb: $A_m \in [2,6]$, $\phi_m \in [30^\circ, 60^\circ]$)

Margins in the Bode diagram



Session outline

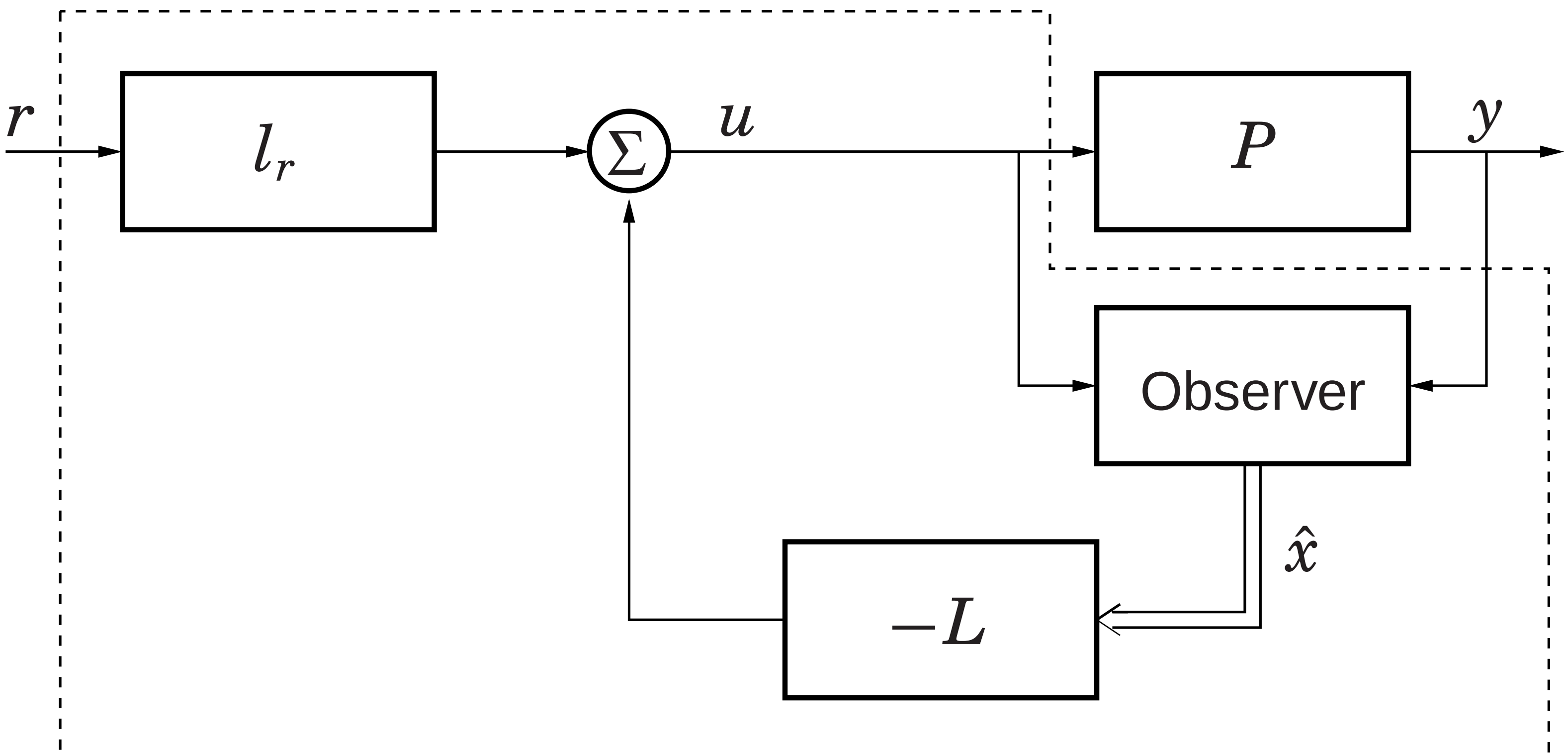
- Feedback and feedforward
- PID Control
- State-space models
- Transfer function models
- Control design using pole placement
- State feedback and observers

Session outline

- Feedback and feedforward
- PID Control
- State-space models
- Transfer function models
- Control design using pole placement
- **State feedback and observers**

State feedback from an observer

A general controller structure that can be applied to systems of any order:



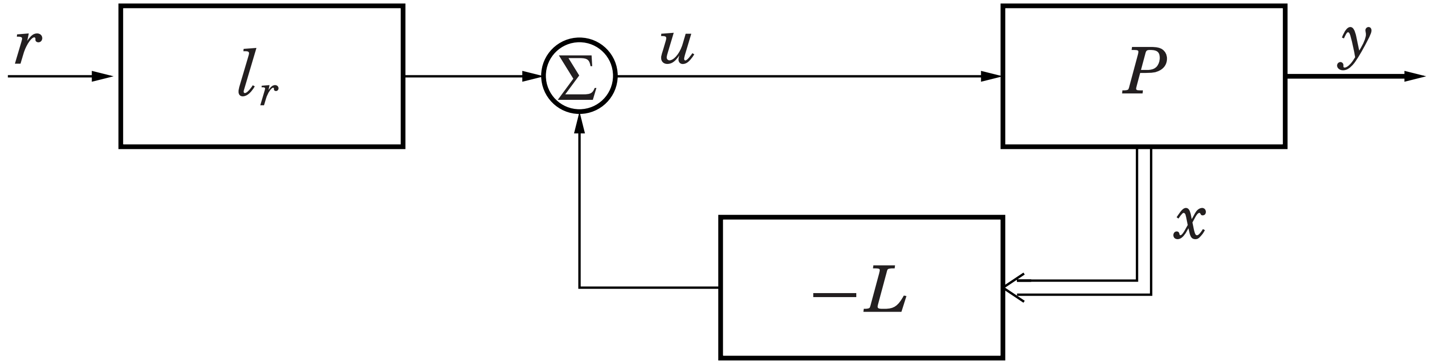
State feedback

Process:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Assume that the full state vector x is measurable. Control law:

$$u = -Lx + l_r r$$



Closed-loop system:

$$\frac{dx}{dt} = (A - BL)x + Bl_r r$$
$$y = Cx$$

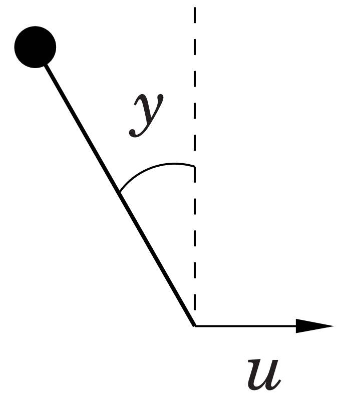
The closed loop poles are given by

$$\det(sI - A + BL) = 0$$

Tuning:

- L is chosen to give the desired poles
- l_r is chosen to give the static gain 1 from r to y

Example - Inverted pendulum



State variables $x_1 = y, \;\; x_2 = \dot{y} \quad \Rightarrow$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Determine a state feedback law (assume $r = 0$)

$$u = -Lx = -\begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

such that the closed-loop characteristic polynomial becomes $s^2 + 1.4s + 1$.

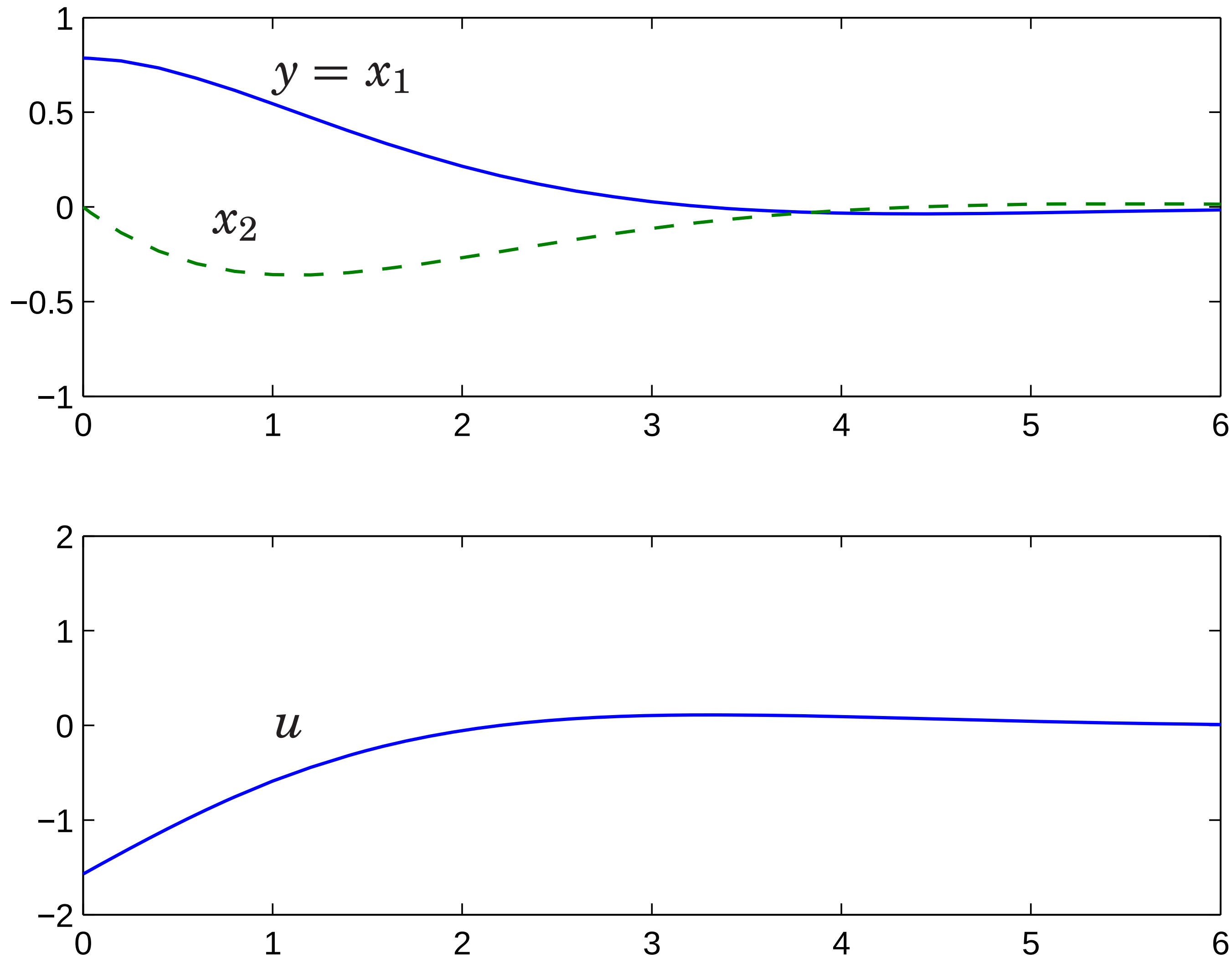
Closed-loop poles:

$$\det(sI - A + BL) = \begin{vmatrix} s & -1 \\ -1 + l_1 & s + l_2 \end{vmatrix} = s^2 + l_2 s - 1 + l_1$$

A comparison with the desired polynomial gives

$$l_1 = 2$$
$$l_2 = 1.4$$

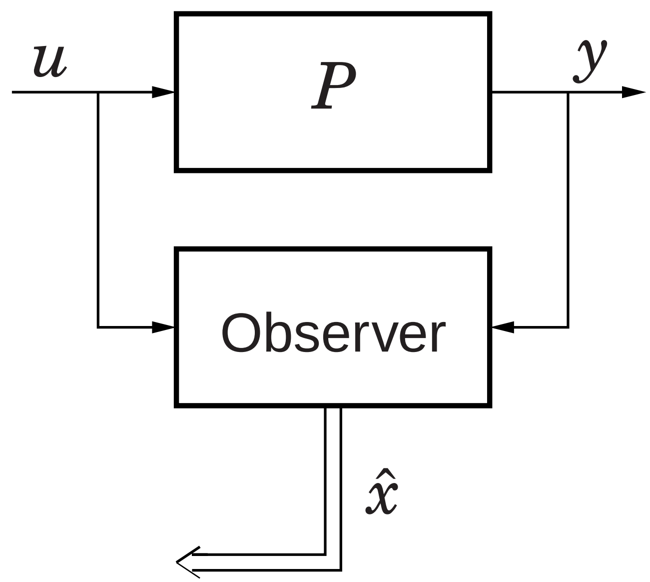
Simulation from $x(0) = [\,0.75 \quad 0\,]^T$:



Observer

It is most often not possible to measure the full state vector x.

The state can then be estimated using an observer:



Observer:

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K\left(y - \hat{y}\right)$$
$$\hat{y} = C\hat{x}$$

Dynamics of the estimation error $\tilde{x} = x - \hat{x}$:

$$\frac{d\tilde{x}}{dt} = Ax + Bu - A\hat{x} - Bu - KC(x - \hat{x}) = (A - KC)\tilde{x}$$

Observer poles:

$$\det(sI - A + KC) = 0$$

Tuning:

- K is chosen to give the desired poles
  - fast poles $\Rightarrow$ fast convergence $\hat{x} \to x$ but sensitive to noise
  - slow poles $\Rightarrow$ slow convergence but robust

Example – Inverted pendulum

Determine an observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K\Big(y - C\hat{x}\Big)$$

with the characteristic polynomial $s^2 + 2.8s + 4$.

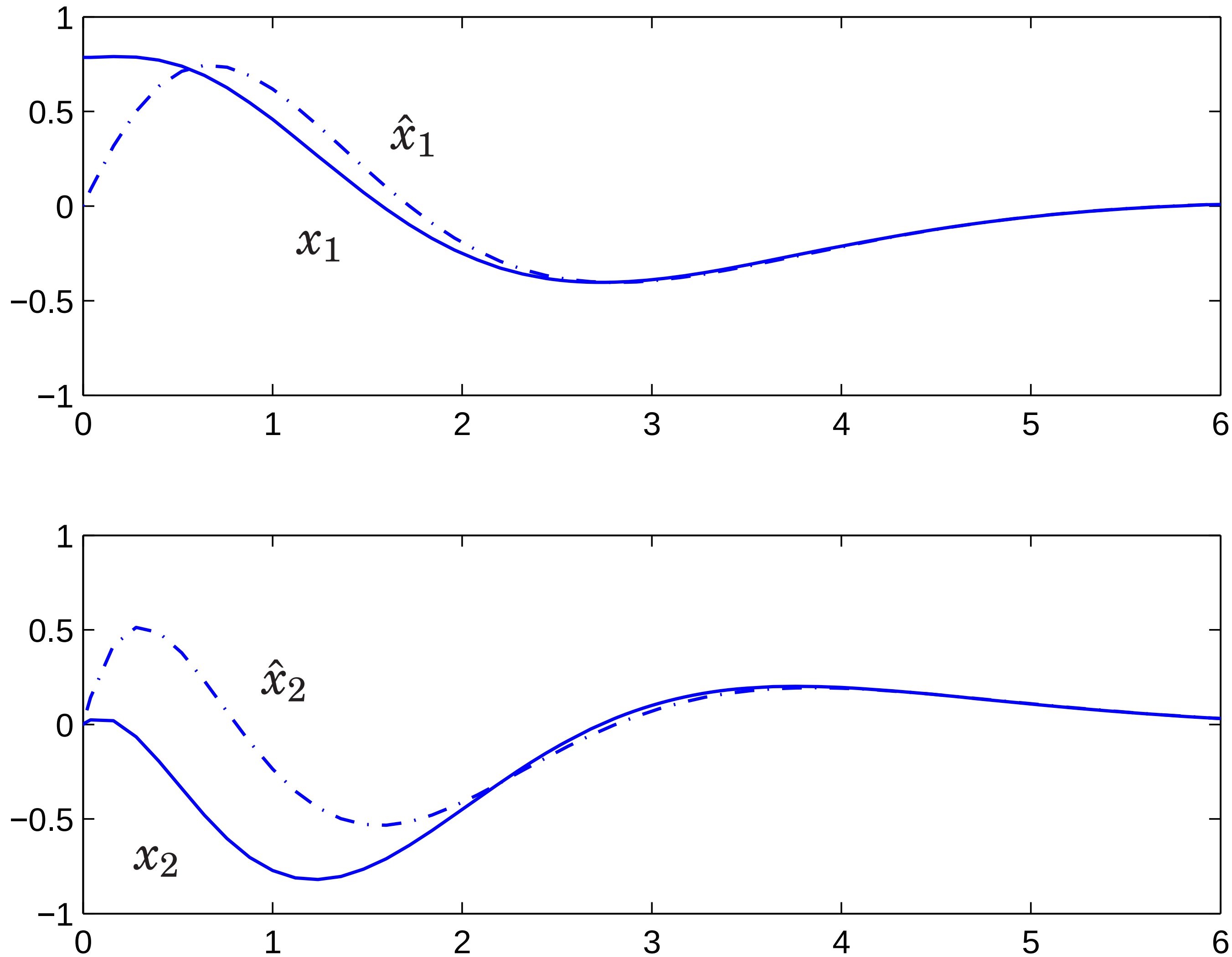
Observer poles:

$$\det(sI - A + KC) = \begin{vmatrix} s + k_1 & -1 \\ -1 + k_2 & s \end{vmatrix} = s^2 + k_1 s - 1 + k_2$$

A comparison with the desired polynomial gives

$$k_1 = 2.8$$
$$k_2 = 5$$

Comparison real–estimated states, $\hat{x}(0) = [\,0 \quad 0\,]^T$:



The complete controller (observer + state feedback) is given by

$$\begin{aligned}\frac{d\hat{x}}{dt} &= A\hat{x} + Bu + K\big(y - C\hat{x}\big)\\ u &= -L\hat{x}\end{aligned}$$

The transfer function of the controller is given by

$$C(s) = -L(sI - A + BL + KC)^{-1}K$$

State feedback from estimated states: