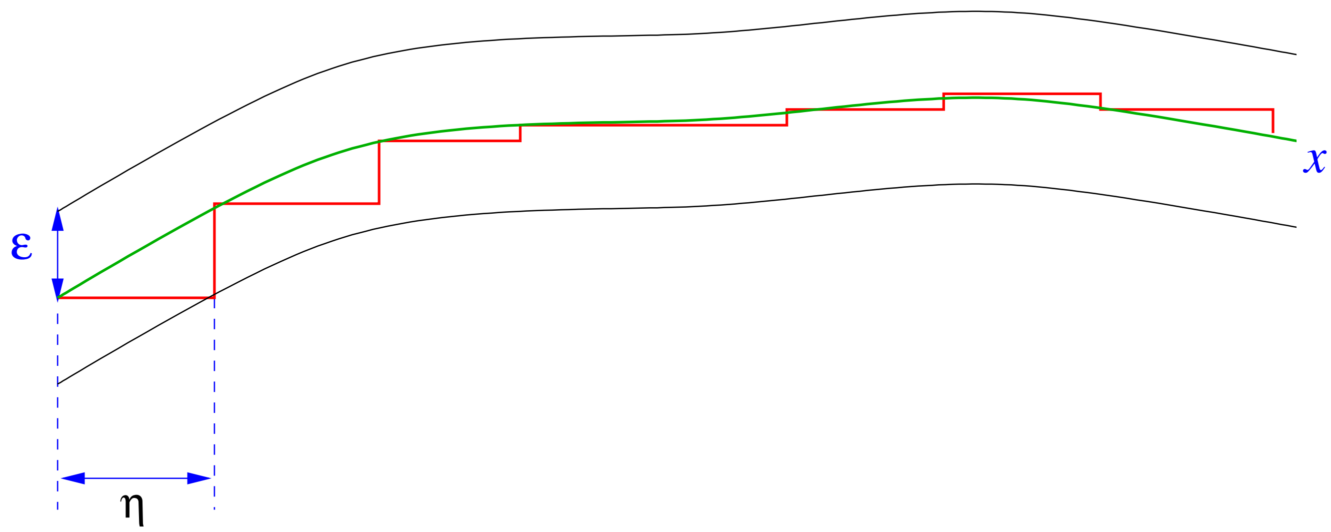
Sampling

Paul Caspi
Laboratoire Verimag (CNRS-UJF-INPG)

- Samplable signals
- Samplable systems
- The error of numerical integration
- The unstable controller case

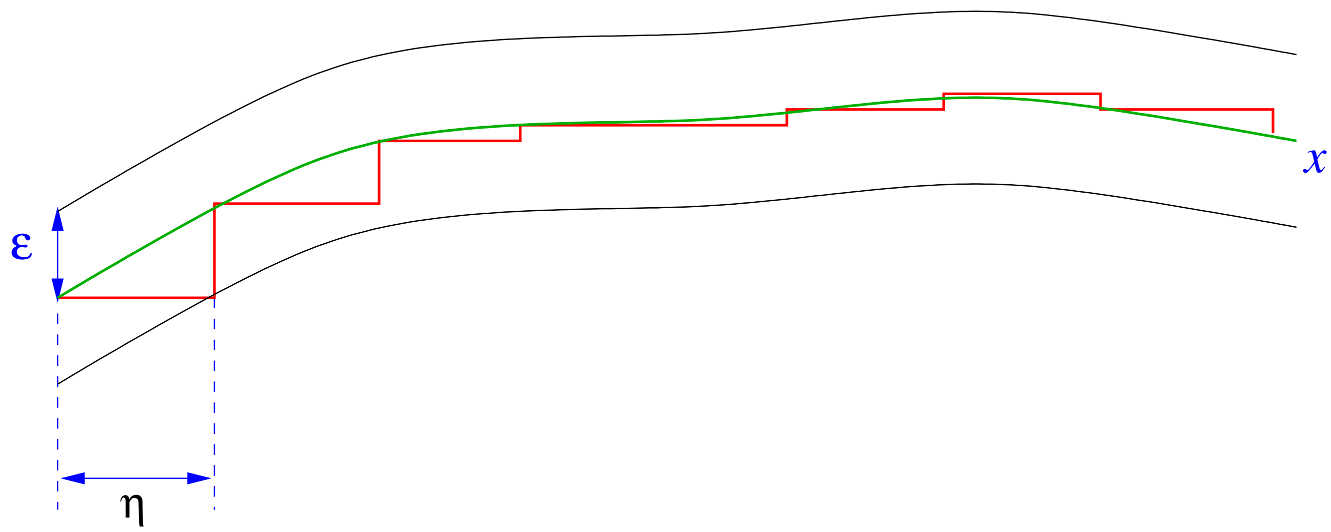
Samplable Signals



$$\begin{cases} \tau \leq \eta \Rightarrow ||x - S_\tau x||_\infty \leq \epsilon \\ \text{where } S_\tau x(t) = x(\lfloor \frac{t}{\tau} \rfloor \tau) \end{cases}$$

What are these good signals that can be sampled with bounded error?

Samplable Signals



$$\begin{cases} \tau \leq \eta \Rightarrow ||x - S_\tau x||_\infty \leq \epsilon \\ \text{where } S_\tau x(t) = x(\lfloor \frac{t}{\tau} \rfloor \tau) \end{cases}$$

What are these good signals that can be sampled with bounded error?

These are the *uniformly continuous* signals

Sampling Continuous Systems

for computer implementation

general idea: forward fixed-step methods

- matches periodic (so-called Time-triggered) computing

two ways :

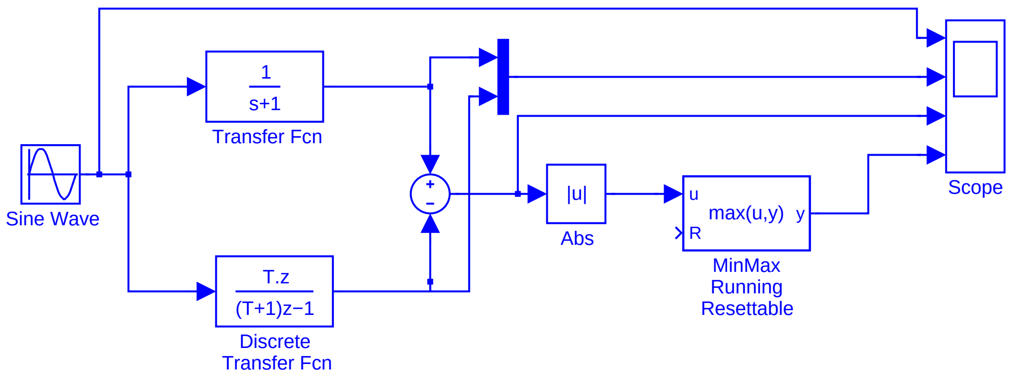
- build a (robust) continuous-time controller and then sample it

- sample the environment to be controlled and design a discrete-time cntroller(sampled-data control systems theory).

In any case one has to choose the sampling period.

Choosing the Sampling Period

To stay consistent with our metric viewpoint, the problem can be stated as follow:



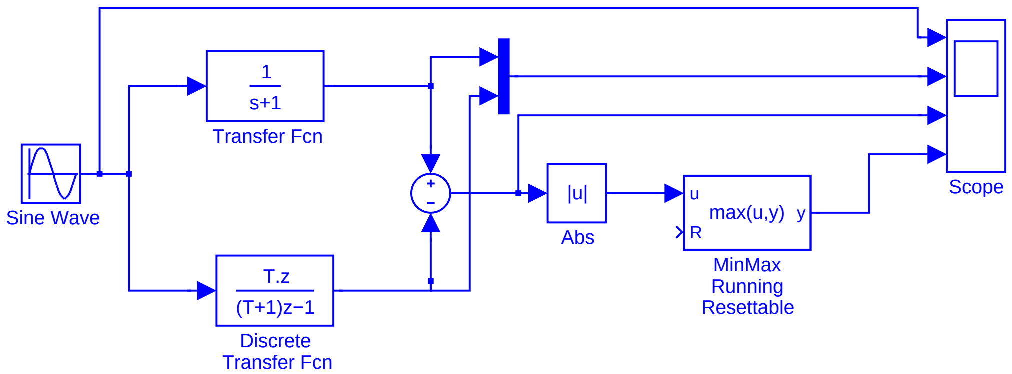
Given any ϵ, can we find a sampling period T such that

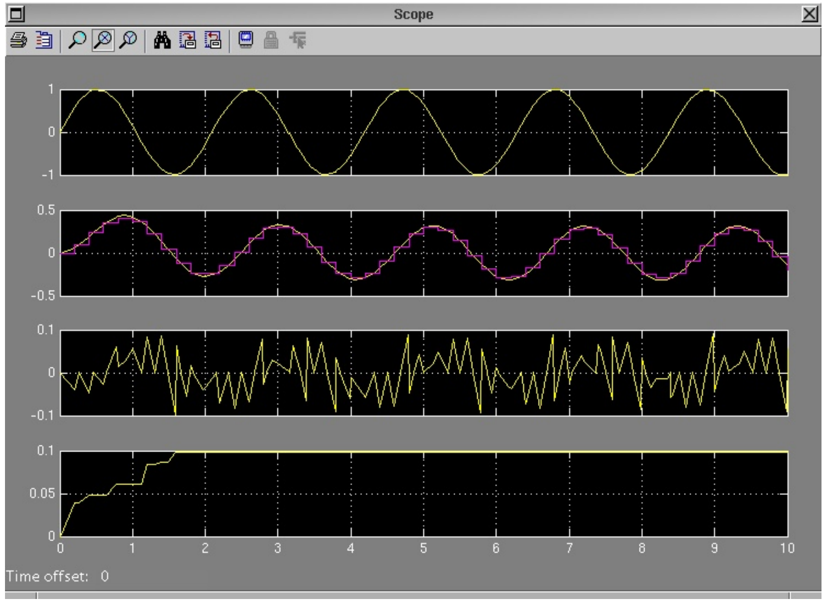
$$||S(x) - S_T(x)||_\infty \leq \epsilon$$

Choosing the Sampling Period

The answer seem positive:

$T = 0.2$





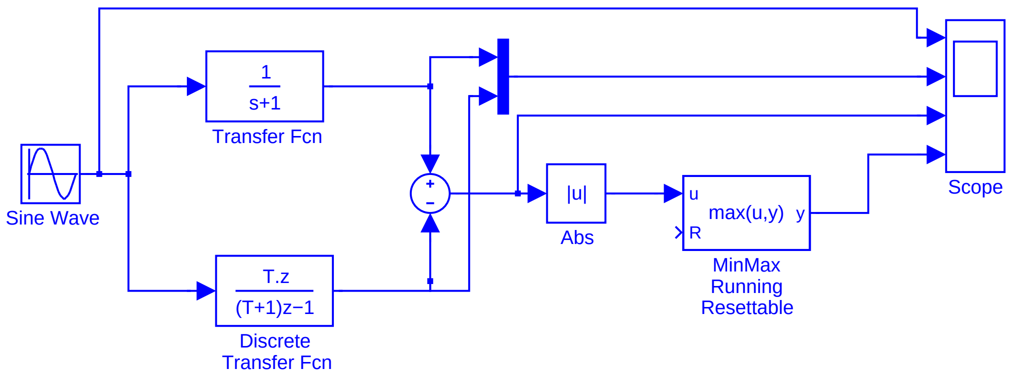
Given any ϵ, can we find a sampling period T such that

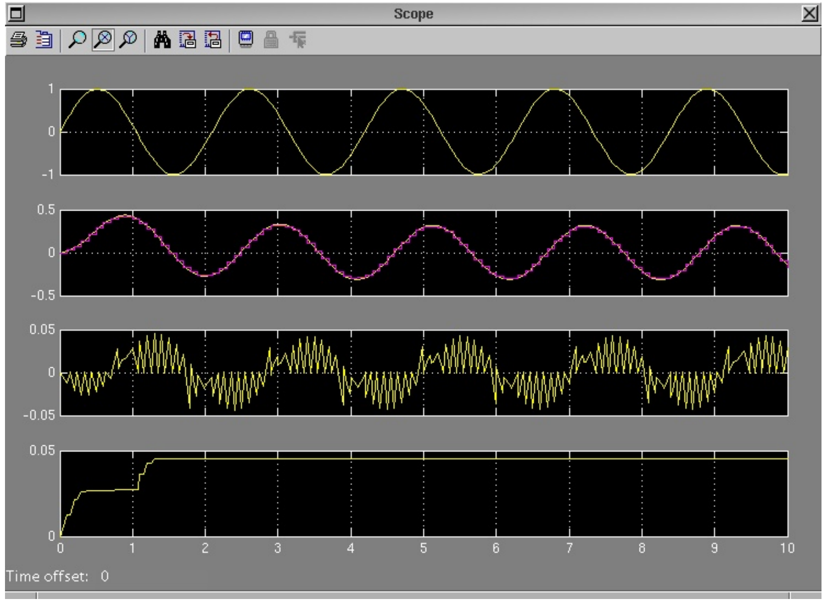
$$||S(x) - S_T(x)||_\infty \leq \epsilon$$

Choosing the Sampling Period

The answer seem positive:

$T = 0.1$





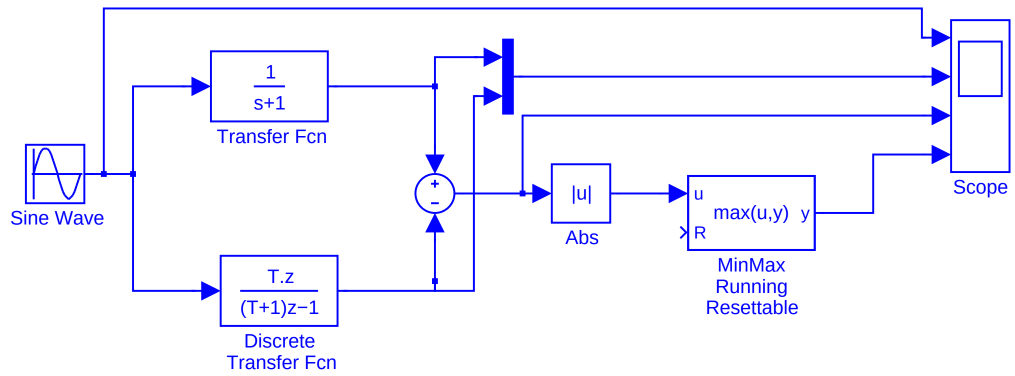
Given any ϵ, can we find a sampling period T such that

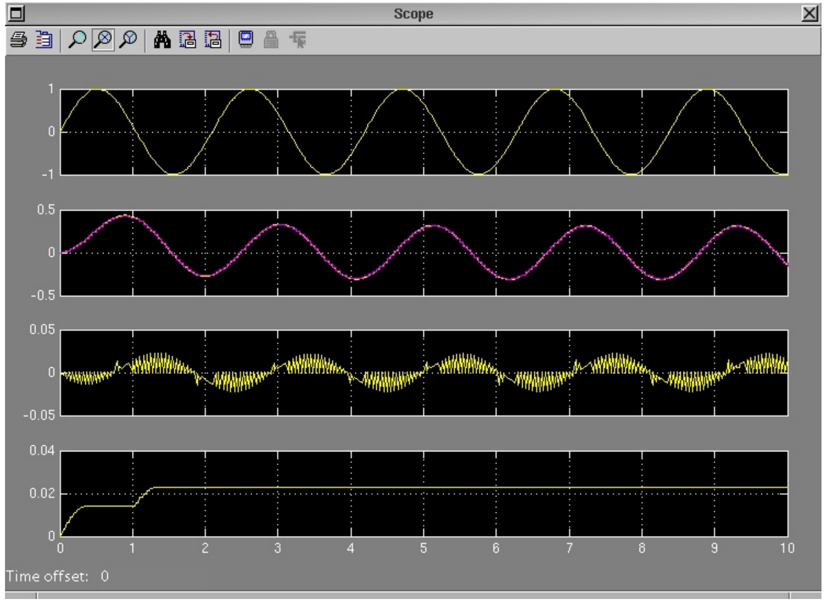
$$||S(x) - S_T(x)||_\infty \leq \epsilon$$

Choosing the Sampling Period

The answer seem positive:

$T = 0.05$





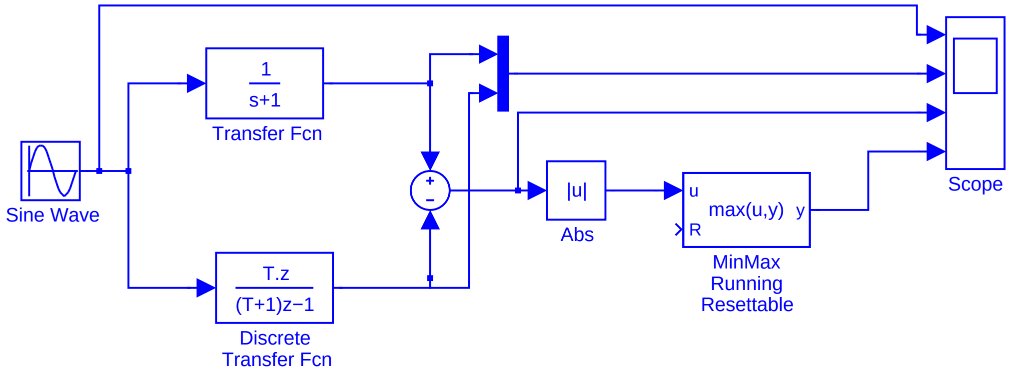
Given any ϵ, can we find a sampling period T such that

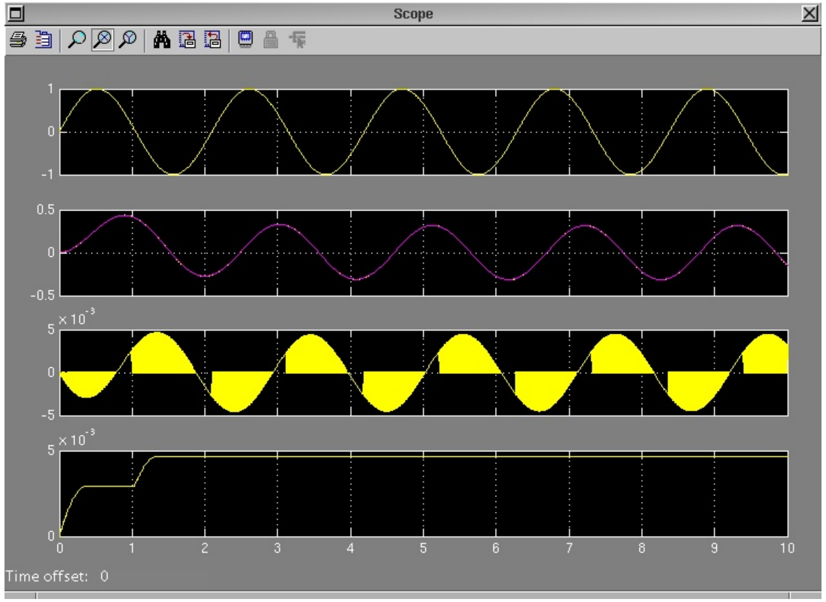
$$||S(x) - S_T(x)||_\infty \leq \epsilon$$

Choosing the Sampling Period

The answer seem positive:

$T = 0.01$





Given any ϵ, can we find a sampling period T such that

$$||S(x) - S_T(x)||_\infty \leq \epsilon$$

How can we Understand It ?

Problem of numerical analysis

Fundamental property : Finite Expansion Theorem

Under some conditions,for all x, t, T, exists $\alpha \in [0, 1]$ such that

$$x(t+T) = \sum_{0}^{k} \frac{T^i}{i!} x^{(i)}(t) + \frac{T^{k+1}}{(k+1)!} x^{(k+1)}(t + \alpha T)$$

How can we Understand It ?

Problem of numerical analysis

Fundamental property : Finite Expansion Theorem

Under some conditions, for all f, u, v, exists $\alpha \in [0,1]$ such that

$$f(v) = \sum_{0}^{k} \frac{(v-u)^i}{i!} f^{(i)}(u) + \frac{(v-u)^{k+1}}{(k+1)!} f^{(k+1)}(\alpha u + (1-\alpha)v)$$

Example : Euler's First Order Method

$$x(t+T) \approx x(t) + Tx'(t)$$

because

$$|x(t+T) - (x(t) + Tx'(t))| \leq \frac{T^2}{2} sup_t |x''(t)|$$

Application

$$x' = f(x, u)$$

yields

$$\hat{x}(t+T) = \hat{x}(t) + T.f(\hat{x}(t), u(t))$$

Accuracy of Euler's Method

$$\hat{x}(t+T) = \hat{x}(t) + T.f(\hat{x}(t), u(t))$$

Question : Can we upperbound the error?

$$e(t) = |x(t) - \hat{x}(t)|$$

This question is critical when using computers for continuous control

Error Decomposition

Let us introduce the fictious signal:

$$\hat{\hat{x}}(t+T) = x(t) + T.f(x(t), u(t))$$

amounting to assuming that we would know exactly the solution $x(t)$ at the previous step (i.e., if we wouldn't have made any previous error)

this allows decomposing the error in two terms:

- integration error $e_i = |x - \hat{\hat{x}}|$

- propagated error $e_p = |\hat{\hat{x}} - \hat{x}|$

and bounding it thanks to triangular inequality:

$$e \leq e_i + e_p$$

Integration Error

We have:

$$x(t+T) - \hat{\hat{x}}(t+T) = x(t+T) - x(t) - Tx'(t)$$

and, by finite expansion, exists $\alpha \in [0, 1]$ such that:

$$x(t+T) - x(t) - Tx'(t) = \frac{T^2}{2} x''(t + \alpha T)$$

Hence the upper bound

$$\boxed{e_i \leq \frac{T^2}{2} \sup_t |x''(t)|}$$

where $x'' = \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial u} u'$

Propagated Error

We have:

$$\hat{x}(t+T) - \hat{\hat{x}}(t+T) = \hat{x}(t) - x(t) + Tf(\hat{x}(t), u(t)) - Tf(x(t), u(t)))$$

Applying finite expansion to the function $x + Tf(x, u)$

Exists $\alpha \in [0, 1]$ such that:

$$\hat{x}(t+T) - \hat{\hat{x}}(t+T) = (\hat{x}(t) - x(t))(1 + T\frac{\partial f}{\partial x}(\alpha\hat{x}(t) + (1-\alpha)x(t), u(t)))$$

Hence

$$\boxed{e_p(t+T) \leq \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x, u)| e(t)}$$

Global Balance

We have:

$$e \leq e_i + e_p$$

$$e_i \leq \frac{T^2}{2} \sup_t |x''(t)|$$

$$e_p(t+T) \leq \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x,u)| e(t)$$

Hence

$$\boxed{e(t+T) \leq \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x,u)| e(t) + \frac{T^2}{2} \sup_t |x''(t)|}$$

Analysis

$$e(t+T) \leq \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x,u)|e(t) + \frac{T^2}{2}\sup_t |x''(t)|$$

Since every thing is positive, the worst case is obtained by the equality

$$e(t+T) = Ae(t) + B$$

with

$$A = \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x,u)| \qquad B = \frac{T^2}{2}\sup_t |x''(t)|$$

The Good Case

$$e(t+T) = Ae(t) + B$$

with

$$A = \sup_{x,u} |1 + T\frac{\partial f}{\partial x}(x,u)| \qquad B = \frac{T^2}{2}\sup_t |x''(t)|$$

- $B < \infty$ and $0 \leq A < 1$

  $\Rightarrow$ e converges toward an upper bound

$$\frac{B}{1-A}$$

Detailling the Good Case

$$A = \sup_{x,u} |1 + T \frac{\partial f}{\partial x}(x,u)| \qquad B = \frac{T^2}{2} \sup_t |x''(t)|$$

$B < \infty$ and $0 \leq A < 1$

Two conditions:

1. $b = \frac{1}{2} \sup_t |x''(t)| < \infty$

2. Exists c, a such that $-c \leq \frac{\partial f}{\partial x}(x,u) \leq -a < 0$

   We then can find T_0 such that for any $T \leq T_0$

$$A = 1 - Ta$$

Detailling the Good Case

Hence:

$$e = \frac{bT^2}{aT} = T\frac{b}{a}$$

hence we can find T small enough to keep this error arbitrarily small.

This is a remarkable result which shows that, on some conditions, one can perform a sequence of numerical approximations in such a way that the successive errors introduced at each step don't accumulate and the overall error stays bounded

Looking at the Conditions

the two conditions are:

1. $b = \frac{1}{2} \sup_t |x''(t)| < \infty$

   the system is stable and the input is bounded and smooth

2. $\frac{\partial f}{\partial x}(x, u)$ is everywhere negative and bounded

   the system is stable and the input is bounded

Influence of stability:

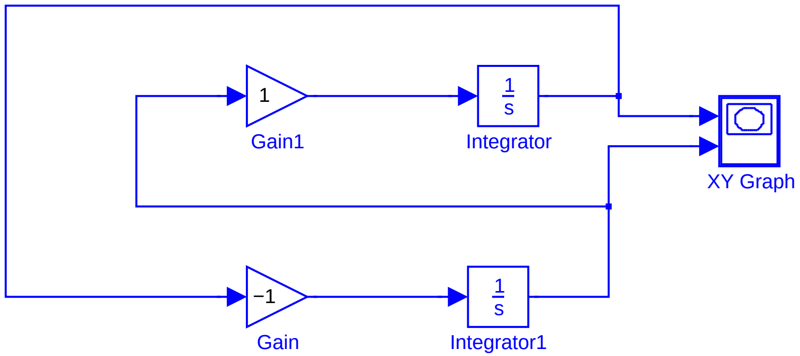
- outputs stay bounded
- previous approximation errors are forgotten

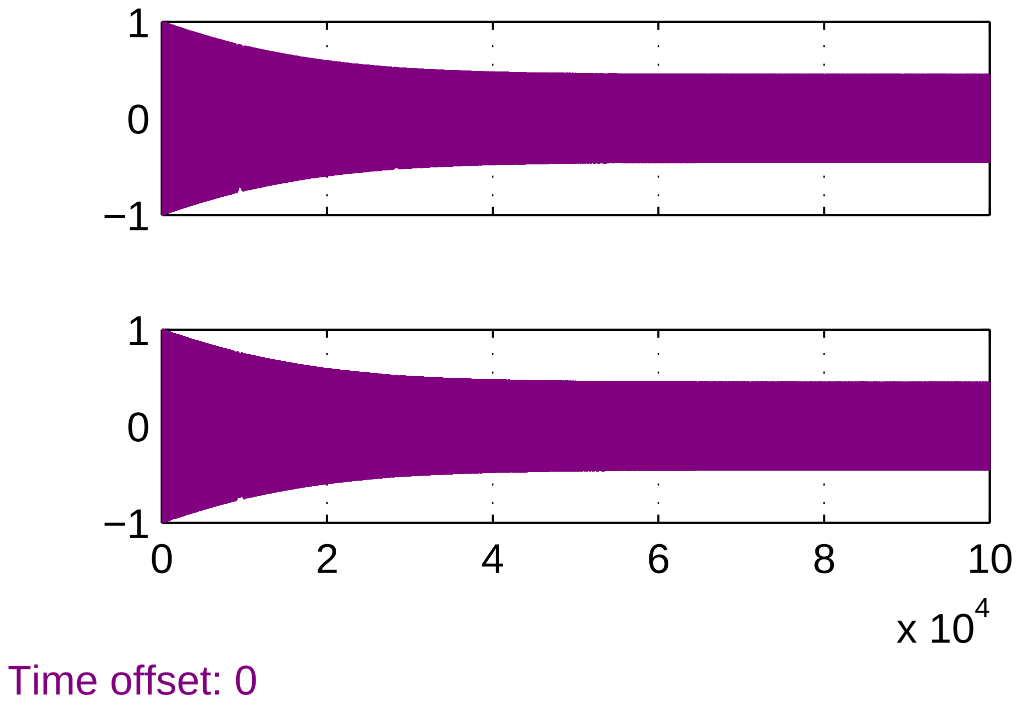
Only stable systems can be implemented on digital computers ?

Example

Sine, cosine : unstable

Long range simulation





Unstable controllers

Unfortunately, this framework is seldom directly applicable because most controllers are not stable. For instance, the integral term of a PID makes it unstable, because the integral accumulates errors without “forgetting” them: the integral term obeys the equation

$$x' = u$$

Thus,

$$\frac{\partial f}{\partial x} = 0$$

and it cannot be upper bounded by a negative constant.

Question

Why then can we use a computer and accurately approximate in discrete time an unstable controller?

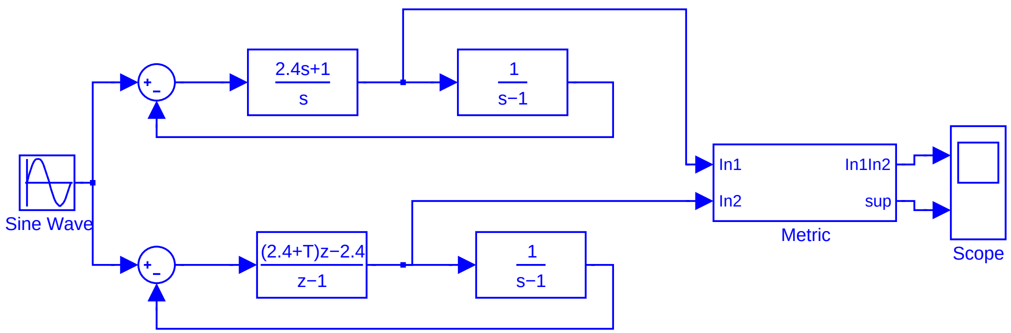
This is a key question for both understanding how controllers operate and how to implement them.

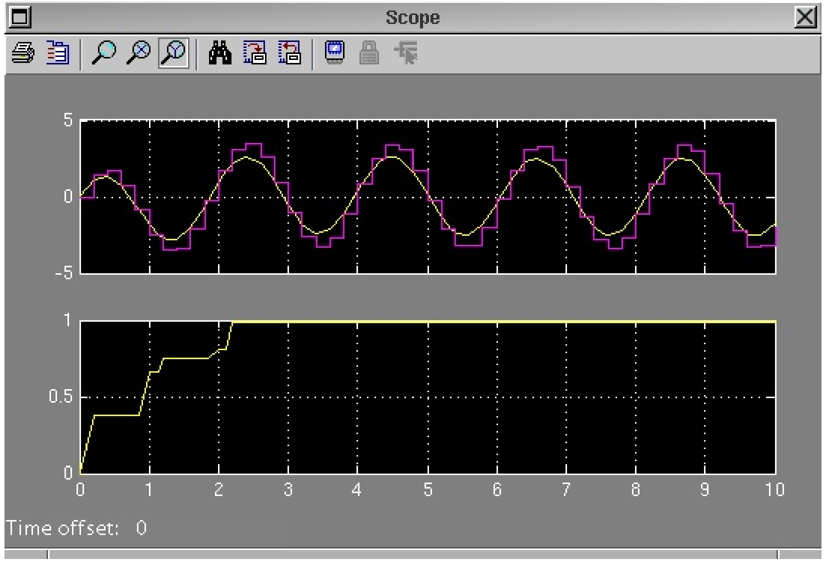
Answer

An unstable controller (like the limit stable PID) can control an unstable system (like the inverse pendulum) in such a way that the overall behaviour of the controlled system be stable!

Example: An unstable control system

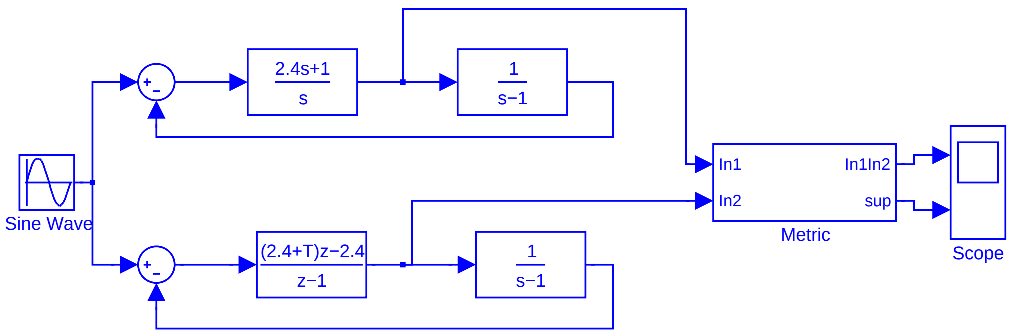
T = 0.2

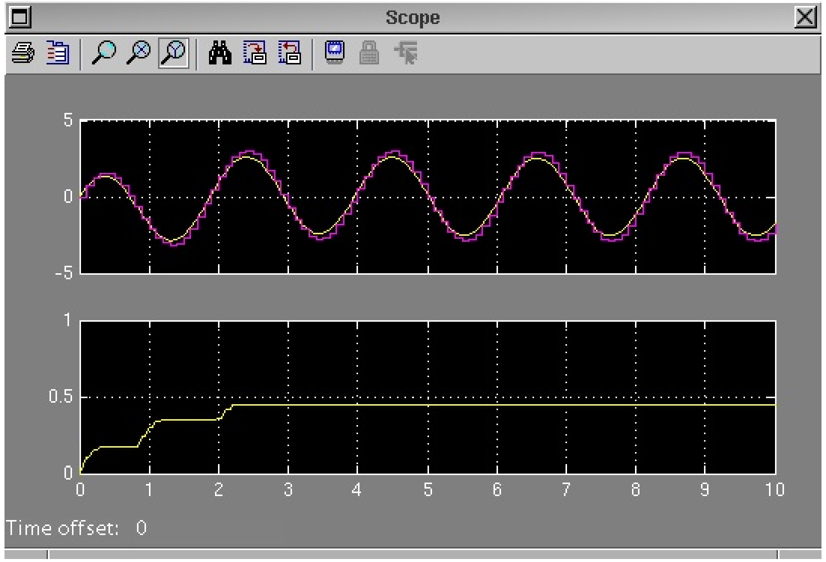




Example: An unstable control system

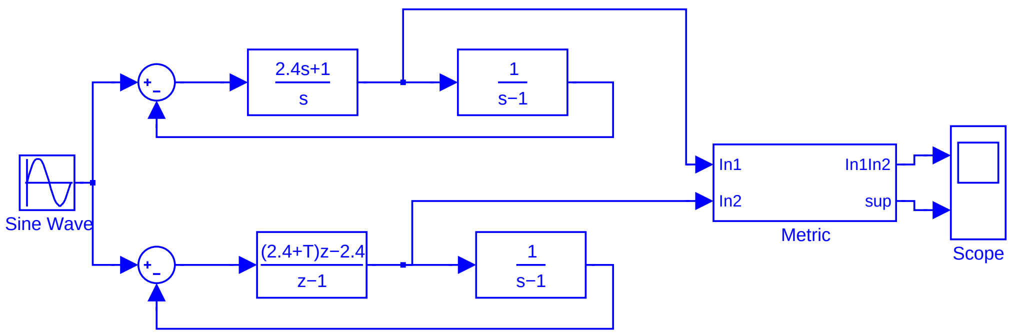
T = 0.1

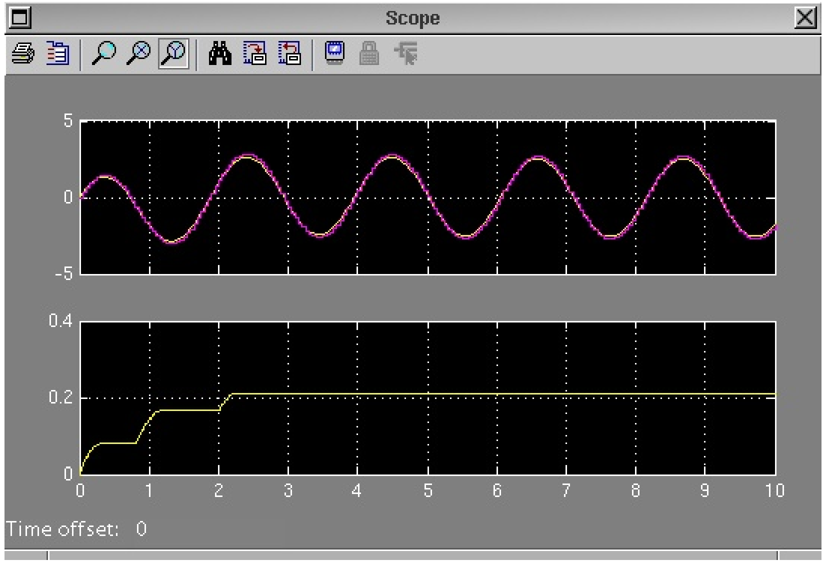




Example: An unstable control system

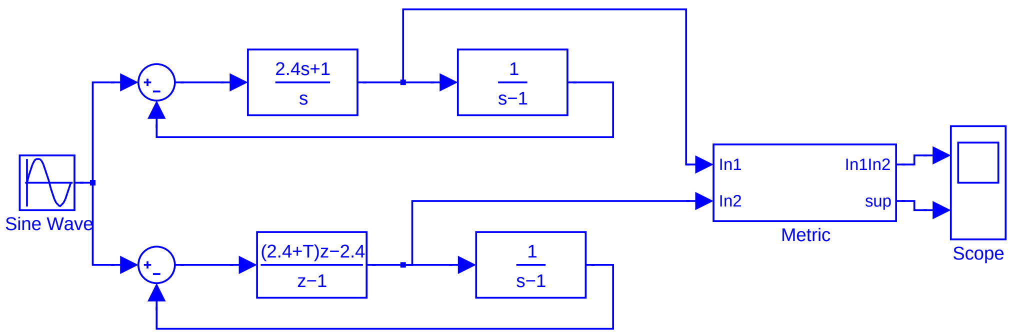
T = 0.05

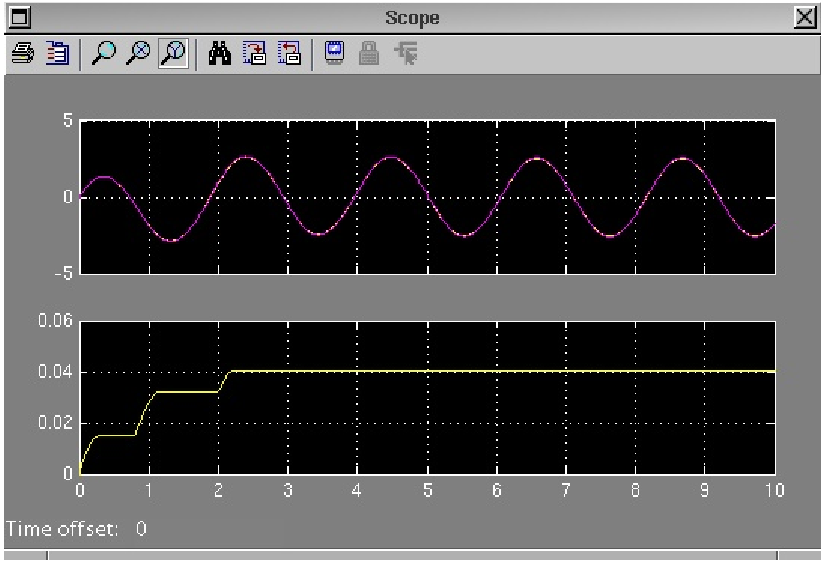




Example: An unstable control system

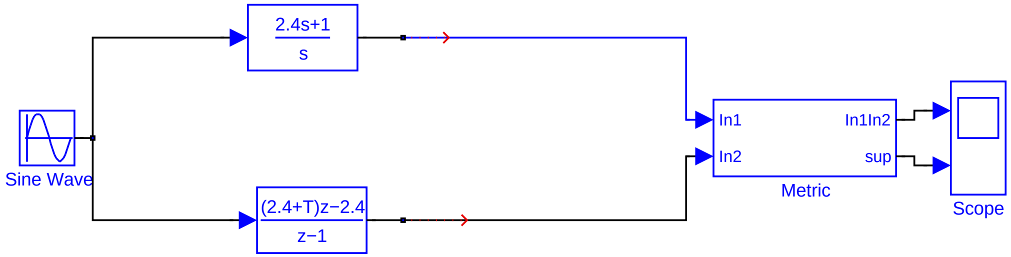
$T = 0.01$

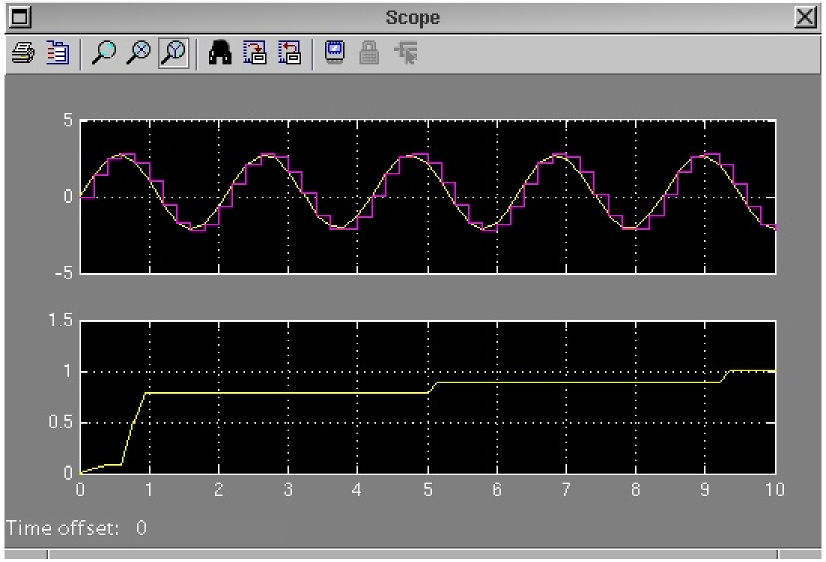




Example: The unstable controller alone

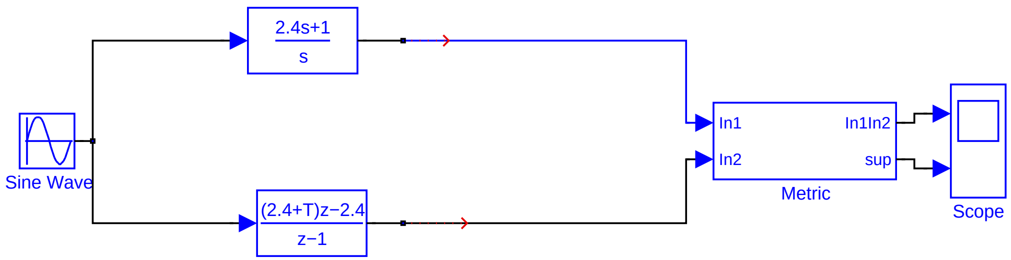
T = 0.2

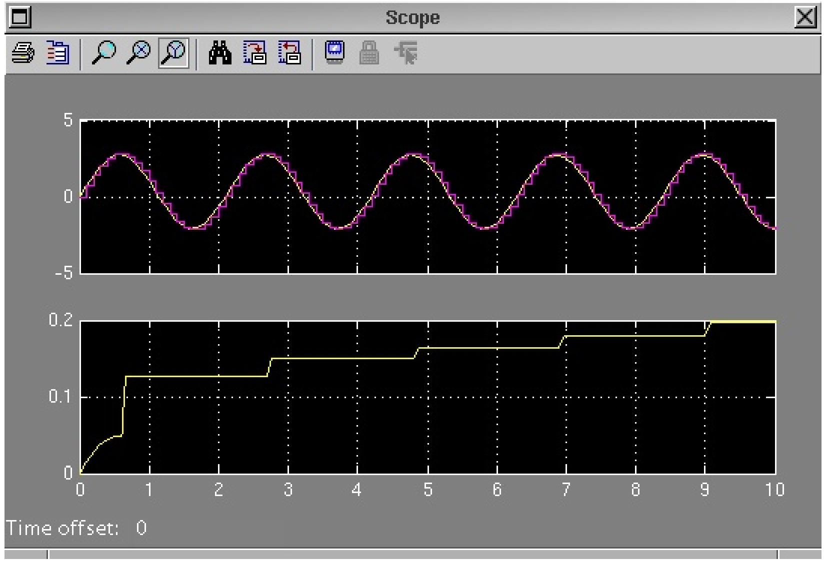




Example: The unstable controller alone

T = 0.1





Example: The unstable controller alone

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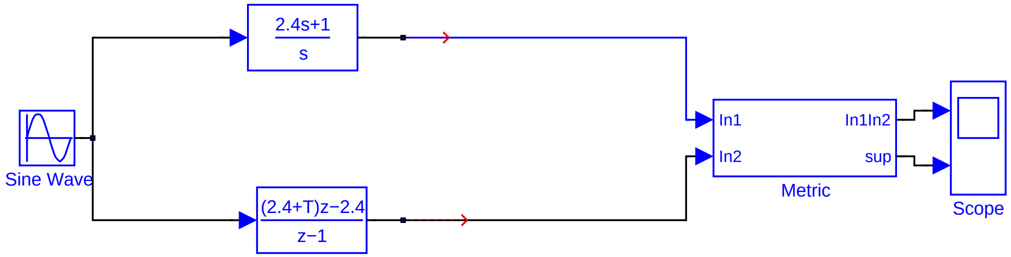

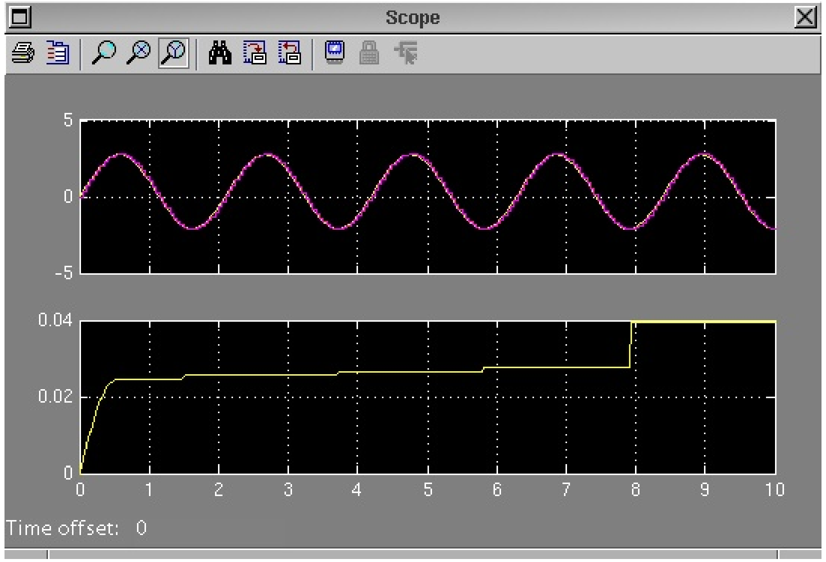

How can we understand it ?

Error bounds in the unstable case

We deal with four systems:

The continuous time controller: $x' = f(x, y, u)$

The environment it controls: $y' = g(x, y)$

The discrete time controller: $\hat{x}(t+T) = \hat{x}(t) + Tf(\hat{x}(t), \hat{y}(t), u)$

The environment it controls: $\hat{y}' = g(\hat{x}, \hat{y})$

We also introduce the fictious

$$\hat{\hat{x}}(t+T) = x(t) + Tf(x(t), y(t), u)$$

and, for the sake of uniformity,

$$\hat{\hat{y}}' = y' = g(x, y)$$

Integration error

It behaves as previously:

$$\begin{aligned}\hat{\hat{x}}(t+T) - x(t+T) &= x(t+T) - x(t) - Tf(x(t), y(t), u(t)) \\ &= \frac{T^2}{2} x''(t + \alpha_x T)\end{aligned}$$

and

$$\hat{\hat{y}}(t+T) - y(t+T) = 0$$

Propagation error

We have

$$\hat{x}(t+T)-\hat{\hat{x}}(t+T) = \hat{x}(t)-x(t)+Tf(\hat{x}(t),\hat{y}(t),u(t))-Tf(x(t),y(t),u(t))$$

but the problem is to treat uniformly x and y. For this, we write, thanks to the finite expansion theorem,

$$\begin{aligned}\hat{y}(t+T)-\hat{\hat{y}}(t+T) = & \quad \hat{y}(t)-y(t)+Tg(\hat{x}(t),\hat{y}(t))-Tg(x(t),y(t)) \\ & + \quad \frac{T^2}{2}(\hat{y}''(t+\alpha_y T)-y''(t+\alpha_y T))\end{aligned}$$

Vector Notation

Let us introduce a

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \quad , \quad F(X,u) = \begin{pmatrix} f(x,y,u) \\ g(x,y) \end{pmatrix}$$

Propagated Error

$$
\begin{aligned}
\hat{X}(t+T) - X(t+T) &= \hat{X}(t) - X(t) + TF(\hat{X}(t), u(t)) - TF(X(t), u(t)) \\
&+ \frac{T^2}{2}(\hat{y}''(t+\alpha_y T) - y''(t+\alpha_y T))
\end{aligned}
$$

Then the finite extension theorem yields

$$
\begin{aligned}
\hat{X}(t+T) - X(t+T) &= [I + T\frac{\partial F}{\partial X}(\alpha\hat{X}(t) + (I-\alpha)X(t), u(t))](\hat{X}(t) - X(t)) \\
&+ \frac{T^2}{2}(\hat{y}''(t+\alpha_y T) - y''(t+\alpha_y T))
\end{aligned}
$$

where I denotes the identity matrix and α is a diagonal matrix with elements in $]0,1[$.

Matrix norm

A vector norm $||\ ||$ extends to a matrix norm by

$$||A|| = \sup_X \frac{||AX||}{||X||}$$

When the Euclidian norm is considered, this amounts to the largest matrix eigen value module.

Overall Error

Then, we get the same maximum error equation as before

$$e(t+T) = Ae(t) + B$$

by taking

$$\begin{aligned} e &= ||\hat{X} - X|| \\ A &= \sup_{X,u} ||I + T\frac{\partial F}{\partial X}(X(t), u(t))|| \\ B &= \sup_{t} \frac{T^2}{2}|x''(t)| + |\hat{y}''(t) - y''(t)| \end{aligned}$$

Stability

The overall system is stable if

$$\sup_{X,u} ||I + T\frac{\partial F}{\partial X}(X(t), u(t))|| < 1$$

Conclusion

If the overall system is stable, the error of the Euler method is bounded by

$$\frac{B}{1 - A}$$

This is an even more remarquable result than the preceeding one because it shows that, though the computations are locally inaccurate (looking at the computer in isolation), they are globally accurate when it comes to the overall behaviour of the system!

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This is an even more remarquable result than the preceeding one because it shows that, though the computations are locally inaccurate (looking at the computer in isolation), they are globally accurate when it comes to the overall behaviour of the system!

But all this is based on continuity. What about discontinuous systems?