Symbolic Methods

Symbolic state-space traversal for finite-state systems

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What you'll learn

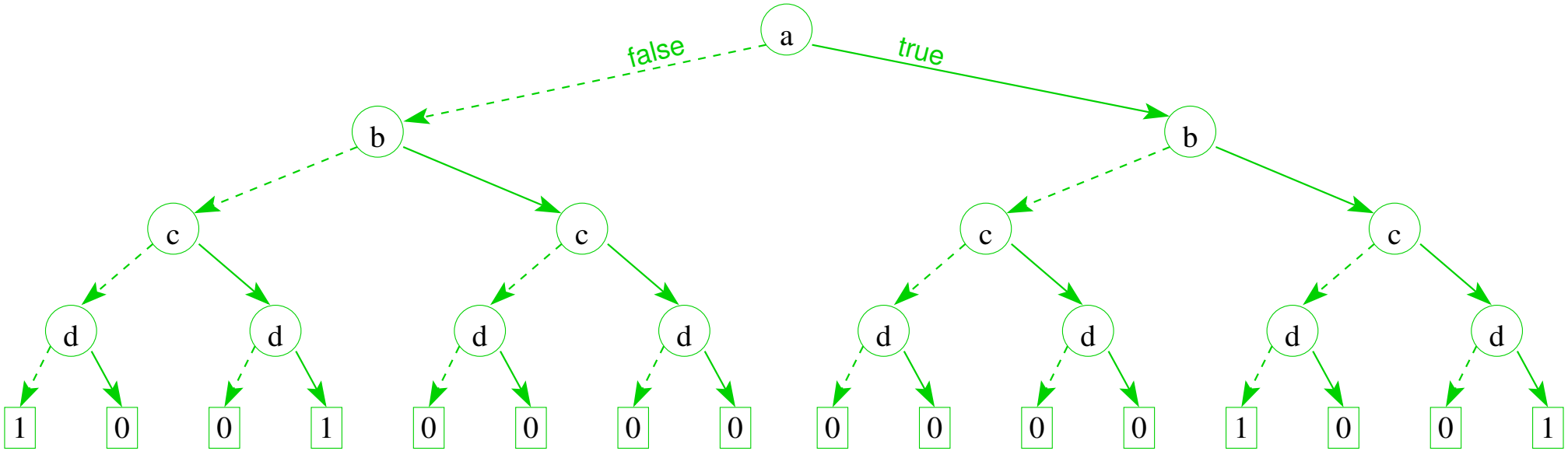
- reduced ordered binary decision diagrams
- symbolic methods for state reachability
  - SAT-based procedures for bounded state reachability
  - full reachability via BDDs
- symbolic CTL model checking

Reduced ordered binary decision diagrams

(RO)BDDs

Binary decision diagrams

An ordered decision tree for $(a \Leftrightarrow b) \wedge (c \Leftrightarrow d)$:



Size exponential in number of variables!

ROBDDs

Obs.: A lot of the tests in the decision diagram are redundant.

Idea: Combine equivalent sub-cases,
i.e. reduce size of the diagram by

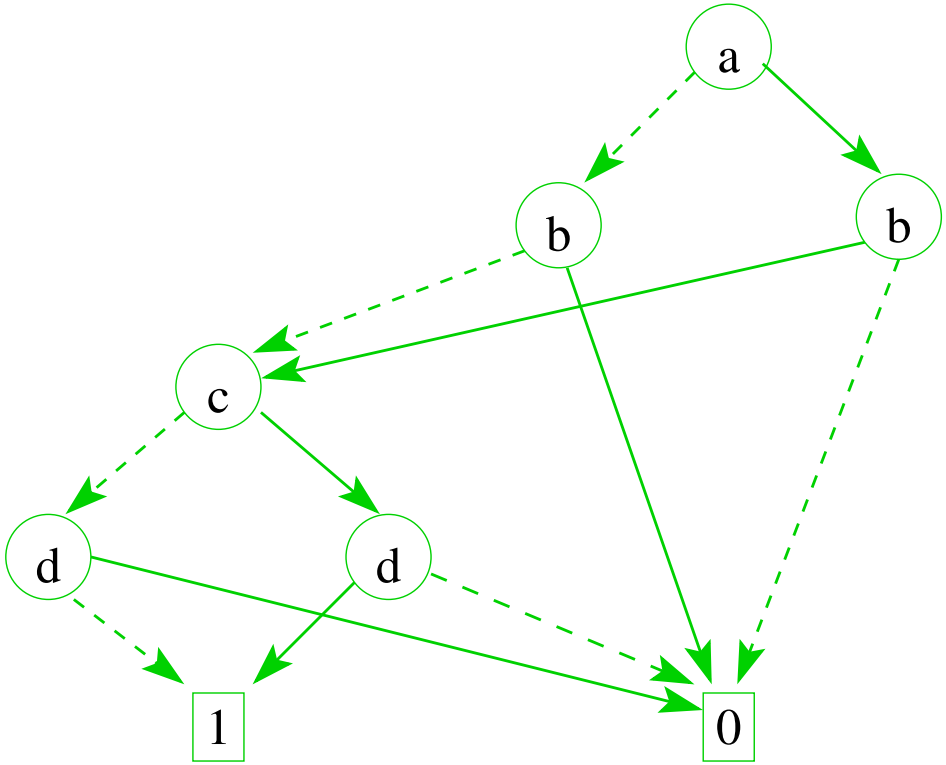
1. omitting nodes that have equivalent left and right sons,
2. sharing common sub-trees:
   - remove duplicate terminal nodes; share instead
   - remove duplicate internal nodes; share instead

Def.: The decision diagrams obtained by above rules are called reduced ordered binary decision diagrams (ROBDDs).

May expect good performance if many substructures are equivalent!

ROBDDs

An ROBDD for $(a \Leftrightarrow b) \wedge (c \Leftrightarrow d)$, using node order $a < b < c < d$:



Note how variable order affects size: Using $a < c < b < d$ would yield a layer with 4 nodes.
For n-bit comparison, we obtain a layer with 2^n nodes if poor order is chosen, yet maximum layer width 2 with appropriate order.

ROBDDS: Some properties

- Given a variable ordering, ROBDDs provide a canonical representation for Boolean functions
  - simple equivalence check, once the ROBDDs have been built:
    - linear in size of BDDs
    - $O(1)$ if sharing across BDDs is used
- Applying a connective to two ROBDDs can be done by simultaneous recursive descent through the two ROBDDs (+acceleration by dynamic programming)
  - $(\text{if } x \text{ then } \phi_t \text{ else } \phi_e) \wedge (\text{if } x \text{ then } \psi_t \text{ else } \psi_e) \equiv$
    $(\text{if } x \text{ then } \phi_t \wedge \psi_t \text{ else } \phi_e \wedge \psi_e)$
  - $\rightarrow$ efficient
  - $\rightarrow$ can construct ROBDDs for non-trivial circuits
- Variable order strongly affects size.
  - need reordering heuristics,
  - even then, some circuits don't permit any good order:
    e.g., multipliers yield exponentially sized BDDs

ROBDD operations

Negation:

Operation: Constructs from an ROBDD B an ROBDD $\mathrm{not}(\mathrm{B})$ with $\mathrm{f}_{\mathrm{not}(\mathrm{B})} = \neg \mathrm{f}_{\mathrm{B}}$, where f_{B} is the truth function encoded by B.

Algorithm: Swap the terminal nodes:

- node 0 is replaced with 1
- node 1 is replaced with 0.

Complexity: $\mathrm{O}(1)$.

ROBDD operations

Boolean junctors:

Operation: Constructs from two ROBDDs B_1, B_2 and a Boolean junctor $\oplus$ an ROBDD $apply(\oplus, B_1, B_2)$ with $f_{apply(\oplus,B_1,B_2)} = f_{B_1} \oplus f_{B_2}$.

Algorithm: Recursively proceed as follows:

- If both B_1 and B_2 are terminal nodes then yield terminal node $f_{B_1} \oplus f_{B_2}$.
- If the top nodes of B_1 and B_2 agree on their variable v then
  1. compute $L = apply(\oplus, left(B_1), left(B_2))$,
  2. compute $R = apply(\oplus, right(B_1), right(B_2))$,
  3. build the OBDD (v, L, R),
  4. reduce it.
- If the top nodes of B_1 and B_2 have different variables v_1, v_2 with $v_1 < v_2$ in the variable order then
  1. compute $L = apply(\oplus, left(B_1), B_2)$,
  2. compute $R = apply(\oplus, right(B_1), B_2)$,
  3. build the OBDD (v, L, R),
  4. reduce it.
- ...

Complexity: $O(|B_1| \cdot |B_2|)$ if memoization is used to save recomputations which may arise due to sharing of subgraphs.

ROBDD operations

Quantification:

Operation: Constructs from an ROBDD B and a variable v an ROBDD $exists(v, \mathrm{B})$ with $\mathrm{f}_{exist(v,\mathrm{B})} = \exists v.\mathrm{f}_{\mathrm{B}}$.

Algorithm:

1. Replace each sub-BDD of B which has a root node n labeled with v by the ROBDD $apply(\vee, left(\mathrm{n}), right(\mathrm{n}))$.
2. Reduce the resulting BDD.

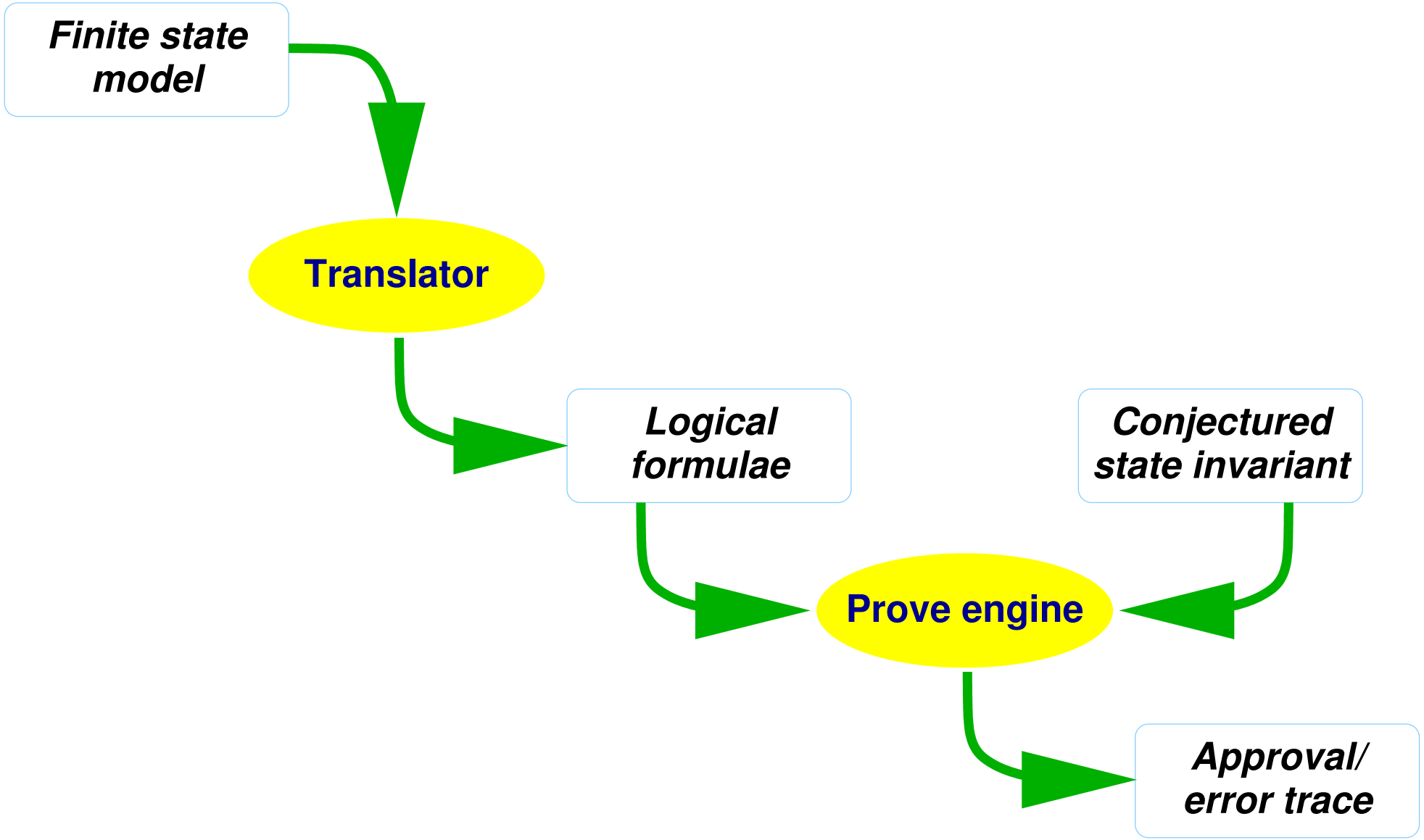
Complexity: $\mathrm{O}(|\mathrm{B}|^2)$.

Note that BDDs obtained by quantifying *multiple* variables may thus grow exponentially in the number of quantified variables.

Symbolic techniques II:

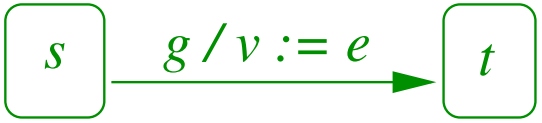
State reachability in finite-state reactive systems

The general framework



Mapping models to formulae (essence of)

- Each control location s is assigned a proposition p_s;
  each symbolic variable v is assigned $\lceil \log_2 |\mathrm{dom}\, v| \rceil$ propositional variables;
- for describing transitions, propositional variables are duplicated:
  - undecorated version encodes pre-state,
  - primed version encodes post-state,



$$\mapsto \quad \phi_{tr} \equiv p_s \wedge \underbrace{[g]}_{\text{proposit. encodings}} \wedge \underbrace{[v' = e]}_{\text{proposit. encodings}} \wedge p'_t$$

$$\mathtt{trans}(x, x') \equiv \bigwedge_{s \text{ state}} \left(p_s \implies \bigvee_{tr \text{ transition from } s} \phi_{tr} \right)$$

- similar for describing initial state set, yielding predicate $\mathtt{init}(x)$.

- Translation can be done componentwise, using conjunction for encoding parallel composition.

$\Rightarrow$ This saves computing the automaton product!

Verification/Falsification

Given: Transition pred. $trans(x, x')$, initial state pred. $init(x)$, conj. invar. $\phi(x)$.

QBF-based algorithm:

1. Start with $R_0(x) = init(x)$.
2. Test for satisfiability of $R_i(x) \wedge \neg\phi(x)$. If test succeeds then report violation of goal.
3. Else build $R_{i+1}(x) = R_i(x) \vee \exists\tilde{x}.\,(R_i(\tilde{x}) \wedge trans(\tilde{x}, x))$.
4. Test whether $R_{i+1}(x) \implies R_i(x)$. If so then report satisfaction of goal. Otherwise continue from step 2, with $i+1$ instead of i.

BF-based algorithm:

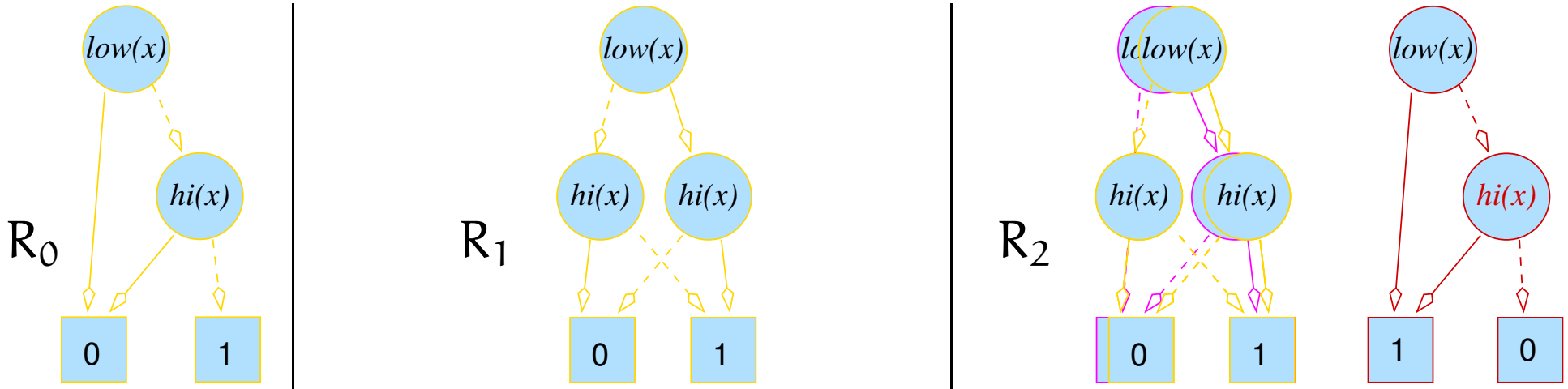
1. For given $i \in \mathbb{N}$ check for satisfiability of
$$\neg \begin{pmatrix} & init(x_0) \wedge trans(x_0, x_1) \wedge \ldots \wedge trans(x_{i-1}, x_i) \\ \Rightarrow & \phi(x_0) \wedge \ldots \wedge \phi(x_i) \end{pmatrix}.$$
If test succeeds then report violation of goal.
2. Otherwise repeat with larger i.

Algorithms by example

Model: $\texttt{VAR}\ x : \{0 \ldots 3\};\ \texttt{INIT}\ x = 0;\ \texttt{NEXT}\ x := 3 - x$

Conjectured Invar.: $\texttt{ALWAYS}\ x = 0$

QBF: BDD-based MC



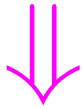
R_0	R_1	R_2
$\overline{l_0} \wedge \overline{h_0} \ \wedge$	$\overline{l_0} \wedge \overline{h_0} \ \wedge\ l_1 = \overline{l_0} \wedge h_1 = \overline{h_0} \wedge$	$\ldots \ \wedge \ldots \wedge\ l_2 = \overline{l_1} \wedge h_2 = \overline{h_1} \wedge$
$(l_0 \vee h_0)$	$(l_0 \vee h_0 \vee \quad l_1 \vee h_1)$	$(\ldots \vee \ldots \vee \quad l_2 \vee h_2)$

BF: SAT-based BMC

Comparison

BDD-based model-checking:

- Normalization within each step of graph coloring.

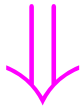


1. Keeps size of intermediate representations compact.
2. Detects saturation of graph coloring.

- Tackles ≈ 500 *state bits*

SAT-based model-checking:

- Purely syntactic expansion, followed by satisfiability check.



- Size of syntactic expansion grows rapidly. E.g. wrt. number of propositional variables used for characterizing n step reachability:

$$\begin{array}{rcl} & \text{statebits} \times (n+1) \\ + & \underbrace{\text{auxbits}}_{>90\%} \times n \end{array}$$

- Tackles $\approx$ 1.000.000 *propositions*, most of which are auxiliary.

[Use cases: verification of high-level models w. limited arithmetic.]

Symbolic methods III:

Beyond reachability

The *pre* operator

Observation: Given

- a predicative encoding S of a state set (with free variables $\vec{x}$),
- a predicative encoding T of the transition relation (with free variables $\vec{x}, \vec{x}'$),

the set $pre(S)$ of states that have a successor in (i.e., satisfying) S can be expressed symbolicly using QBF operators:

$$pre(S) = \exists \vec{x}'.T \wedge S[\vec{x}'/\vec{x}]$$

This can be used for determining all sequential predecessors of a whole set of states in one sweep, thus implementing predecessor colouring "in parallel".

Symbolic CTL model checking

Using the *pre* operator, CTL model checking can be performed by any QBF engine, e.g. by BDDs:

Formula	**Algorithm**	**Result**
propos. P	return $[P]$	Formula f_P denoting P-states
$\mathsf{EX}\,\phi$	return $\mathit{pre}(f_\phi)$	Formula $f_{\mathsf{EX}\phi}$ denoting all states satisfying $\mathsf{EX}\,\phi$
$\mathsf{EG}\,\phi$	Incrementally build $S_0 = f_\phi$ $S_{i+1} = f_\phi \wedge \mathit{pre}(S_i)$ until $(S_n \iff S_{n+1})$ holds	Formula $f_{\mathsf{EG}\phi} = S_n$ denoting all states satisfying $\mathsf{EG}\,\phi$
$\phi\,\mathsf{EU}\,\psi$	Incrementally build $S_0 = f_\psi$ $S_{i+1} = f_\psi \vee (f_\phi \wedge \mathit{pre}(S_i))$ until $(S_n \iff S_{n+1})$ holds	Formula $f_{\phi\mathsf{EU}\psi} = S_n$ denoting all states satisfying $\phi\,\mathsf{EU}\,\psi$

If I characterizes initial states then $I \implies f_\phi$ is to be checked finally.