CounterExample-Guided Abstraction Refinement (CEGAR)

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(with many slides © S. Ratschan[b])

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The problem

- Abstraction is a powerful method for verifying systems
  - maps complex system (e.g., infinite state) to simpler system (e.g., finite Kripke structure)
  - simpler model may be amenable to automatic state-exploratory verification
- but finding the right abstraction is hard
  - may be too coarse $\leadsto$ verification fails
  - may be too fine $\leadsto$ state-space exploration impossible
  - may even be too fine in some places and too coarse in others

The idea

In manual verification, we often add information on demand:

- Upon a failing proof, we analyze the reasons and
- add preconditions as necessary.

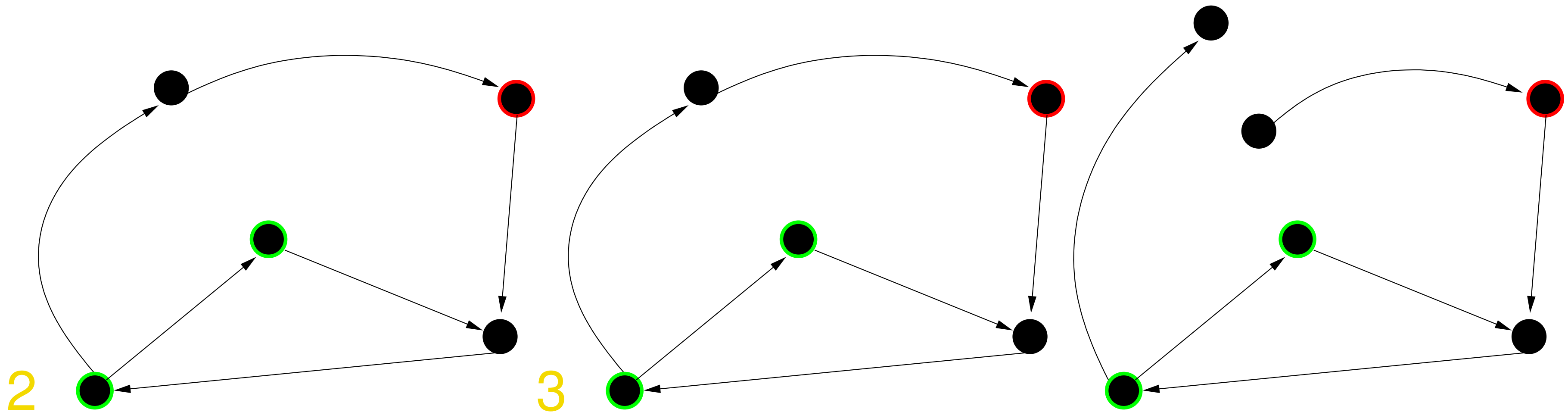
Can we do the same within abstraction-based model checking?

- Upon a failing proof, let the model-checker analyze the reasons and
- refine the abstraction as necessary.

Abstraction Refinement

Idea:

- conservatively approximate the hybrid system by a finite Kripke structure (the *abstraction*)



- if abstraction safe, done
- while abstraction not safe, refine it
- counter-example based: refine to remove a given spurious counter-example (Clarke et al. 03, Alur et al. 03)

Basic CEGAR

Spurious counterexample

Def: Let $A \succ C$ be an homomorphic abstraction wrt. abstraction function h. Let ϕ be an $\forall$CTL formula and $\pi = (c_1, c_2 \ldots)$ be an anchored path of C witnessing violation of ϕ on C.
Then π is called a counterexample for ϕ on C.
Furthermore, $h(\pi) = (h(c_1), h(c_2), \ldots)$ then is an anchored path of A which violates ϕ, i.e. a counterexample on A. We do then call $h(\pi)$ the abstract counterexample corresponding to π and we call π the concrete counterexample corresponding to $h(\pi)$.

Def: If π_A is a counterexample on the abstraction $A \succ C$ which has no corresponding concrete counterexample on C then we call π_A a spurious counterexample.

Abstraction Refinement

Def: If $C \prec A' \prec A$ then A and A' are called abstraction of C and A' is called an abstraction refinement of A.

Idea: Whenever there is a spurious counterexample in A, identify an abstraction refinement A' that lacks that particular spurious counterexample.

CEGAR algorithm (simple version: invariants)

To verify $C \models AGp$ do

1. build finite Kripke structure $A \succ C$,
2. model-check $A \models AGp$,
3. if this holds then report $C \models AGp$ and stop,
4. otherwise validate the counterexample on C, i.e., find a corresponding concrete counterexample,
5. if a corresponding concrete counterexample exists then report $C \not\models AGp$ and stop,
6. otherwise use the spurious counterexample to refine A and restart from 2.

The crucial ingredients of CEGAR

- Model checking,
- validation/concretization of counterexample,
- guided refinement of abstraction.

Validation of counterexample

Given: $A \succ C$ and an abstract counterexample $\phi = (a_1, a_2, \dots, a_n)$ on A.

Alg: Provided we can effectively manipulate pre-images of the abstraction morphism h, proceed as follows:

1. Compute $S_1 := h^{-1}(a_1) \cap I_C$, where I_C is the set of initial states of C,
2. For $i = 2$ to n, compute $S_i := h^{-1}(a_i) \cap \mathrm{Post}(S_{i-1})$.
   Abort as soon as some S_i becomes $\emptyset$.
   In this case, the counterexample has been shown to be spurious.
3. In case of proper termination of the loop, the counterexample is real.

N.B. Assumes that $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test)!

State splitting

Idea: For a set $C_i = h^{-1}(a_i)$ of concrete states represented by an abstract state a_i occurring in the spurious counterexample, split it into $C_i \cap \mathrm{Post}(h^{-1}(a_{i-1}))$ and $C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1}))$, provided both non-empty (or into $C_1 \cap I_C$ and $C_1 \setminus I_C$ in case $i = 1$).

Approach: Replace a_i by two states a_i^+ and a_i^- representing $C_i \cap \mathrm{Post}(h^{-1}(a_{i-1}))$ and $C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1}))$, resp.

Technique: Replace the Kripke structure $A = (V, E, L, I)$ by $A' = (V', E', L', I')$ with

- $V' = V \setminus \{a_i\} \cup \{a_i^+, a_i^-\}$, where the latter are $\notin V$,
- $$E' = E \cap (V' \times V') \cup \{(a_i^+, a_i^-), (a_i^-, a_i^+)\} \cup \{(a, a_i^+) \mid (a, a_i) \in E\} \cup \{(a, a_i^-) \mid (a, a_i) \in E, a \neq a_{i-1}\} \cup \{(a_i^+, a), (a_i^-, a) \mid (a_i, a) \in E\}$$
- $$L'(v) = \begin{cases} L(v) & \text{if } v \in V, \\ L(a_i) & \text{if } v \in \{a_i^+, a_i^-\}, \end{cases}$$
- $$I' = \begin{cases} I & \text{if } C_i \cap I_C = \emptyset, \\ I \setminus \{a_i\} \cup \{a_i^+\} & \text{otherwise.} \end{cases}$$

Resulting morphism

$$h'(c) = \begin{cases} a_i^+ & \text{if } c \in C_i \cap \mathrm{Post}(h^{-1}(a_{i-1})), \\ a_i^- & \text{if } c \in C_i \setminus \mathrm{Post}(h^{-1}(a_{i-1})), \\ h(c) & \text{otherwise.} \end{cases}$$

Refining E': transition pruning

Observation: Pre- and post-images of $h'^{-1}(a_i^+)$ or $h'^{-1}(a_i^-)$ may well have empty intersections with sets that the pre- or post-set of $h'^{-1}(a_i)$ did intersect with.
In such cases, E' contains spurious edges.

Solution: Remove such edges by pruning E' to

$$E'' = \{(s,t) \in E' \mid \mathrm{Post}(h'^{-1}(s)) \cap h'^{-1}(t) \neq \emptyset\}$$

CEGAR algorithm (simple version: invariants)

To verify $\mathcal{C} \models AGp$ do

1. build finite Kripke structure $\mathcal{A} \succ \mathcal{C}$,
2. model-check $\mathcal{A} \models AGp$,
3. if this holds then report $\mathcal{C} \models AGp$ and stop,
4. otherwise validate the counterexample on $\mathcal{C}$, i.e., find a corresponding concrete counterexample,
5. if a corresponding concrete counterexample exists then report $\mathcal{C} \not\models AGp$ and stop,
6. otherwise use the spurious counterexample to *split states in* $\mathcal{A}$,
7. *perform transition pruning* on the resulting refinement $\mathcal{A}'$,
8. goto 2.

Concrete version is just an example, variants of split/prune rules abound.

Application to hybrid systems

- Above procedure is effective if $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test).
- This is in general not true for hybrid systems.

$\Rightarrow$ Need to embed an appropriate form of approximation of the above sets into CEGAR.

CEGAR on hybrid states

Conservative approximation of state sets

Application to hybrid systems

- "Naive" CEGAR procedure is effective if $h^{-1}(a_i)$, $\mathrm{Post}(S_i)$, and their intersections are computable (in the sense of an effective emptiness test).
- In general not true for hybrid systems, thus embed an appropriate form of approximation of the above sets into CEGAR.
- Main difficulty is computation of successor states: explicit (jumps) and implicit transitions (flows, defined by ODE)
  - Multiple shapes of overapproximation can be used
    - various effective representations of subsets of $\mathbb{R}^n$: rectangular boxes, zonotopes, polyhedra, ellipsoids, ...,
    - multiple techniques for conservatively approximating hybrid transitions (jumps & flows)
  - can be combined to obtain an adaptive CEGAR algorithm
    - e.g., proceeds from coarse to fine, investing computational effort to increase precision when necessary.

Computing successors

- CEGAR algorithm applies different approximations of successor computation in sequence,
- proceeds from coarse to fine, investing more computational effort to increase precision only when necessary,
- hope is that crucial deductions (absence of counterexamples, non-concretizability of a certain counter-example) can often be obtained on coarse abstractions,
- CEGAR needs to compute different relative successors $\mathit{Succ}(X, Y) = \mathit{Post}(X) \cap Y$, where $X, Y \in \mathcal{P}(\mathbb{R}^n)$.
- Can approximate these by any operation $\mathit{SUCC} : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ with
  1. Overapproximation: $\mathit{SUCC}(X, Y) \supseteq \mathit{Post}(X) \cap Y$,
  2. Reasonability: $\mathit{SUCC}(X, Y) \subseteq Y$.

Validation of counterexample

Given: $A \succ C$ and an abstract counterexample $\phi = (a_1, a_2, \ldots, a_n)$ on A.

Alg: For a sequence of successively tighter overapproximations $(SUCC_i)_{i=1,\ldots,k}$, proceed as follows:

1. Start with $i = 1$, i.e., the coarsest approximation.
2. Compute $S_1^i := \mathrm{overapprox}_i(h^{-1}(a_1) \cap I_C)$, where I_C is the set of initial states of C,
3. For $j = 2$ to n, compute $S_j^i := SUCC_i(S_{j-1}^i, h^{-1}(a_i))$
   Abort as soon as some S_j^i becomes $\emptyset$.
   In this case, the counterexample is spurious.
4. In case of proper termination of the inner loop, restart at 1. with $i := i + 1$, i.e., the next finer approximation, if $i < k$.
5. If the inner loop terminates regularly for $i = k$, then the abstract counterexample can't be refuted by any of the overapproximations. (Probably is real.)

HSolver

Overapproximation via Constraint-based Reasoning

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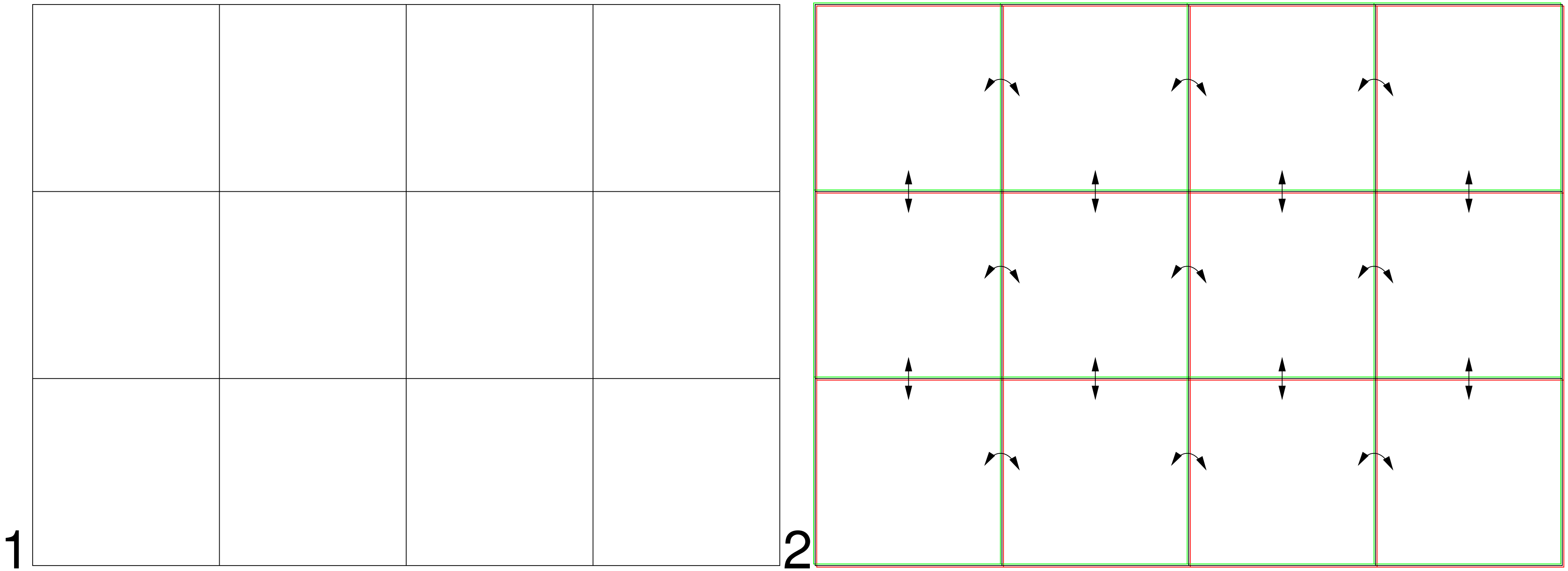
Starting Point: Interval Grid Method

Stursberg/Kowalewski et. al., one-mode case:



$[-5,-1]4\dot{x} \in [-5,1]$

- put transitions between all neighboring hyperrectangles (*boxes*), mark all as initial/unsafe
- remove impossible transitions/marks (interval arithmetic check on boundaries/boxes)

Result: finite abstraction

Interval arithmetic

Is a method for calculating an interval *covering* the possible values of a real operator if its arguments range over intervals:

$$\begin{aligned}
[a, A] \overset{\circ}{+} [b, B] &= [a + b, A + B] \\
[a, A] \overset{\circ}{\cdot} [b, B] &= [\min\{ab, aB, Ab, AB\}, \max\{ab, aB, Ab, AB\}] \\
\overset{\circ}{\min}\,([a, A], [b, B]) &= [\min\{a, b\}, \min\{A, B\}] \\
\overset{\circ}{\sin}\,([a, A]) &= \begin{bmatrix} \min\{\sin x \mid x \in [a, A]\}, \\ \max\{\sin x \mid x \in [a, A]\} \end{bmatrix} \\
\overset{\circ}{f}\,([a, A], [b, B], \ldots) &= \begin{bmatrix} \min\{f(\vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\}, \\ \max\{f(\vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\} \end{bmatrix}
\end{aligned}$$

Theorem: For each term t with free variables $\vec{v}$:

$$\{t(\vec{v} \mapsto \vec{x}) \mid \vec{x} \in [a, A] \times [b, B] \times \ldots\} \subseteq \overset{\circ}{t}\,(v_1 \mapsto [a, A], v_2 \mapsto [b, B], \ldots)$$

Is the approximation tight?

1. In the limit: **yes!**

$$t(\vec{v} \mapsto \vec{x}) = \overset{\circ}{t}\,(v_1 \mapsto [x_1, x_1], v_2 \mapsto [x_2, x_2], \ldots)$$

$$t(\vec{v} \mapsto \vec{x}) = \lim_{\varepsilon \to 0} \overset{\circ}{t}\,(v_1 \mapsto [x_1 - \varepsilon, x_1 + \varepsilon], v_2 \mapsto [x_2 - \varepsilon, x_2 + \varepsilon], \ldots)$$

provided t is uniformly continuous.

2. In general: **No!** If $a < A$ then

$$x - x(x \mapsto [a, A]) = [a, A] \overset{\circ}{-} [a, A] = [a - A, A - a] \neq [0, 0]$$

Dependency problem of interval arithmetic:

Tight bounds only if each variable occurs at most once!

Interval Grid Method II

Check safety on resulting finite abstraction

if safe: finished, otherwise: refine grid;
continue until success

More modes: separate grid for each mode

Jumps: also check using interval arithmetic

Discussion

Advantages:

- can deal with constants that are only known up to intervals
- interval tests cheap (e.g., compare to explicit computation of continuous reach sets, or full decision procedures)

Disadvantages:

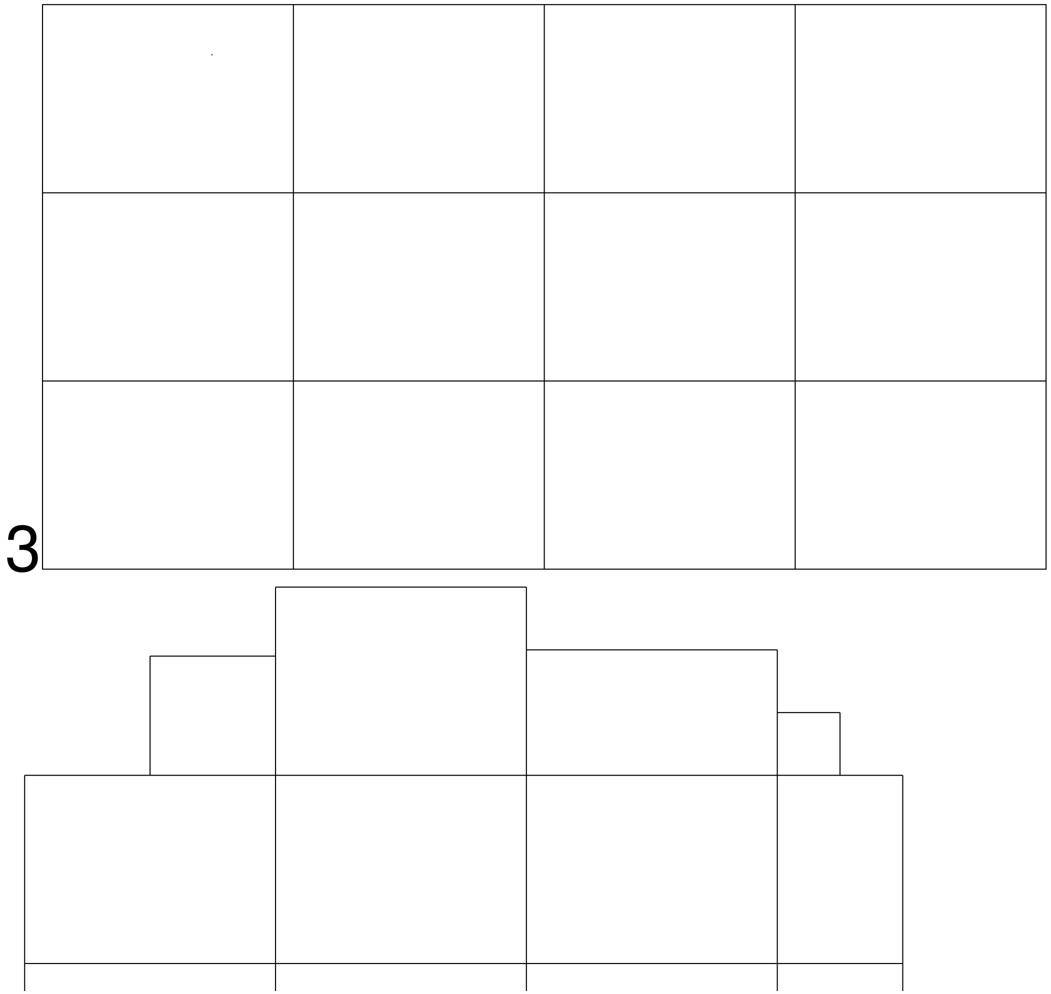
- may require a very fine grid to provide an affirmative answer (curse of dimensionality)
- ignores the continuous behavior within the grid elements

Let's remove them!

Removing Disadvantages

reflect more information in abstraction without creating more boxes by splitting

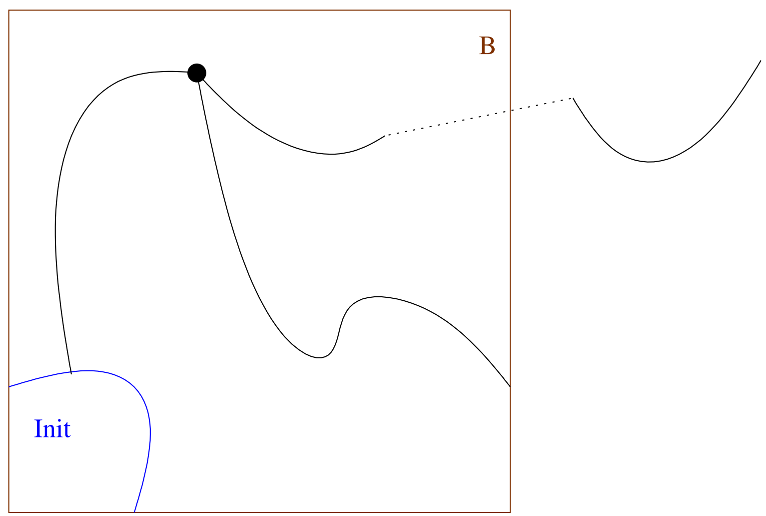
Observation: we do not need to include information on unreachable state space, remove such parts from boxes



Reach Set Pruning

A point in a box B can be reachable

- from the initial set via a flow in B
- from a jump via a flow in B
- from a neighboring box via a flow in B



formulate corresponding constraints, remove all points from box that do not fulfill one of these constraints

Constraints in Specification

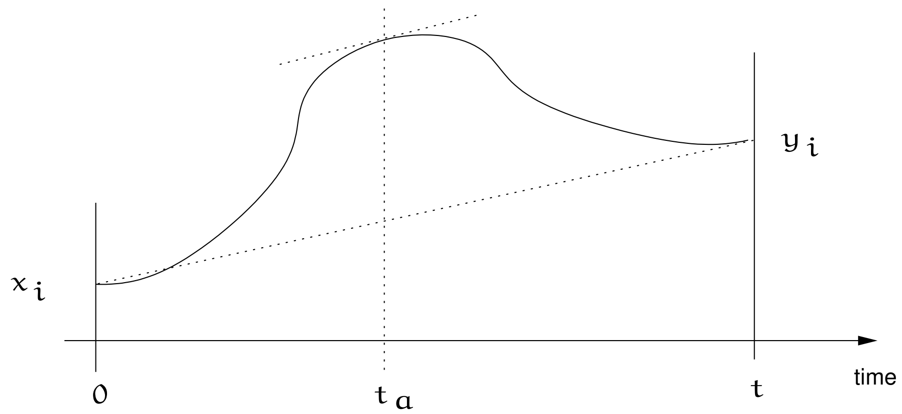
We specify system using constraints:

- $\mathrm{Flow}(s, \vec{x}, \dot{\vec{x}})$ (e.g., $s = \mathtt{off} \rightarrow \dot{x} = x \sin(x) + 1 \quad \ldots)$
  - ˙purely syntactic!
  - even implicit and algebraic!
- $\mathrm{Jump}(s, \vec{x}, s', \vec{x}')$ (e.g.,
  $(s = \mathtt{off} \wedge x \geq 10) \;\rightarrow\; (s' = \mathtt{on} \wedge x' = 0))$
- $\mathrm{Init}(s, \vec{x})$

Reachability Constraints

Lemma (n-dimensional mean value theorem): For a box B, mode s, if a point $(y_1, \ldots, y_n) \in B$ is reachable from a point $(x_1, \ldots, x_n) \in B$ via a flow in B then

$$\exists t \in \mathbb{R}_{\geq 0} \bigwedge_{1 \leq i \leq n} \exists a_1, \ldots, a_k, \dot{a}_1, \ldots, \dot{a}_k[(a_1, \ldots, a_k) \in B \wedge$$
$$Flow(s, (a_1, \ldots, a_k), (\dot{a}_1, \ldots, \dot{a}_k)) \wedge y_i = x_i + \dot{a}_i \cdot t]$$



Denote this constraint by $flow_B(s, \vec{x}, \vec{y})$.

Reachability Constraints

Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, if $\vec{y} \in B$ is reachable from the initial set via a flow in B then

$$\exists \vec{x} \in B\,[\mathit{Init}(s, \vec{x}) \wedge \mathit{flow}_B(s, \vec{x}, \vec{y})]$$

Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, $\vec{y} \in B$, $(s, \vec{y})$ is reachable from a jump from a box B^* and mode s^* via a flow in B then

$$\exists \vec{x}^* \in B^* \exists \vec{x} \in B\,[\mathit{Jump}(s^*, \vec{x}^*, s, \vec{x}) \wedge \mathit{flow}_B(s, \vec{x}, \vec{y})]$$

Reachability Constraints

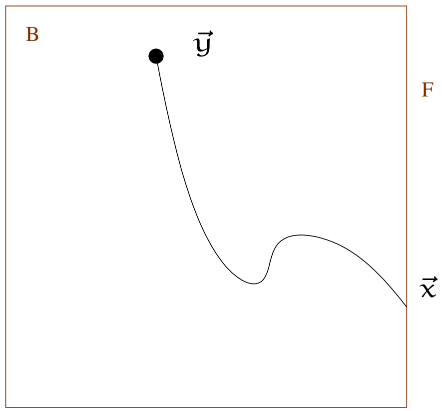
Lemma: For a box $B \subseteq \mathbb{R}^k$, mode s, if $\vec{y} \in B$ is reachable from a neighboring box over a face F of B and a flow in B then

$$\exists \vec{x} \in F\left[\mathrm{incoming}_F(s, \vec{x}) \wedge \mathrm{flow}_B(s, \vec{x}, \vec{y})\right],$$

where $\mathrm{incoming}(s, \vec{x})$ is of the form

$$\exists \dot{x}_1, \ldots, \dot{x}_k[\mathrm{Flow}(s, \vec{x}, (\dot{x}_1, \ldots, \dot{x}_k)) \wedge \dot{x}_j \; r \; 0]$$

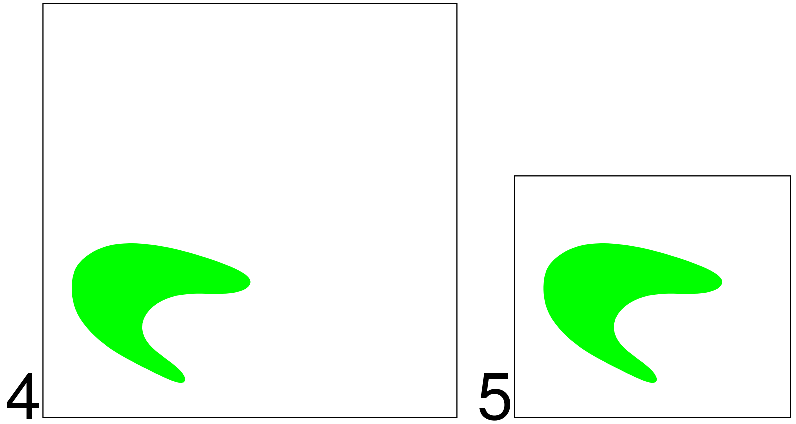
where $r \in \{\leq, \geq\}$, $j \in \{1, \ldots, k\}$ depends on the face F



for corners etc. a little bit more involved

Using Constraints

After substituting definitions, getting rid of quantifiers, interval constraint propagation algorithms can remove parts from boxes not fulfilling such constraints.



- correct handling of rounding errors
- almost negligible time
- result not necessarily tight (but tight for $\mathtt{flow}_{\mathrm{B}}(s, \vec{x}, \vec{y})$ in linear case)

`http://rsolver.sourceforge.net`