Second Lecture: Basics of model-checking for finite and timed systems

Jean-François Raskin
Université Libre de Bruxelles
Belgium

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Plan of the talk

- Labelled transition systems
- Properties of labeled transition systems: Reachability - Safety - Büchi properties
- Pre-Post operators
- Partial orders - Fixed points
- Symbolic model-checking
- Application to TA: region equivalence, region automata, zones

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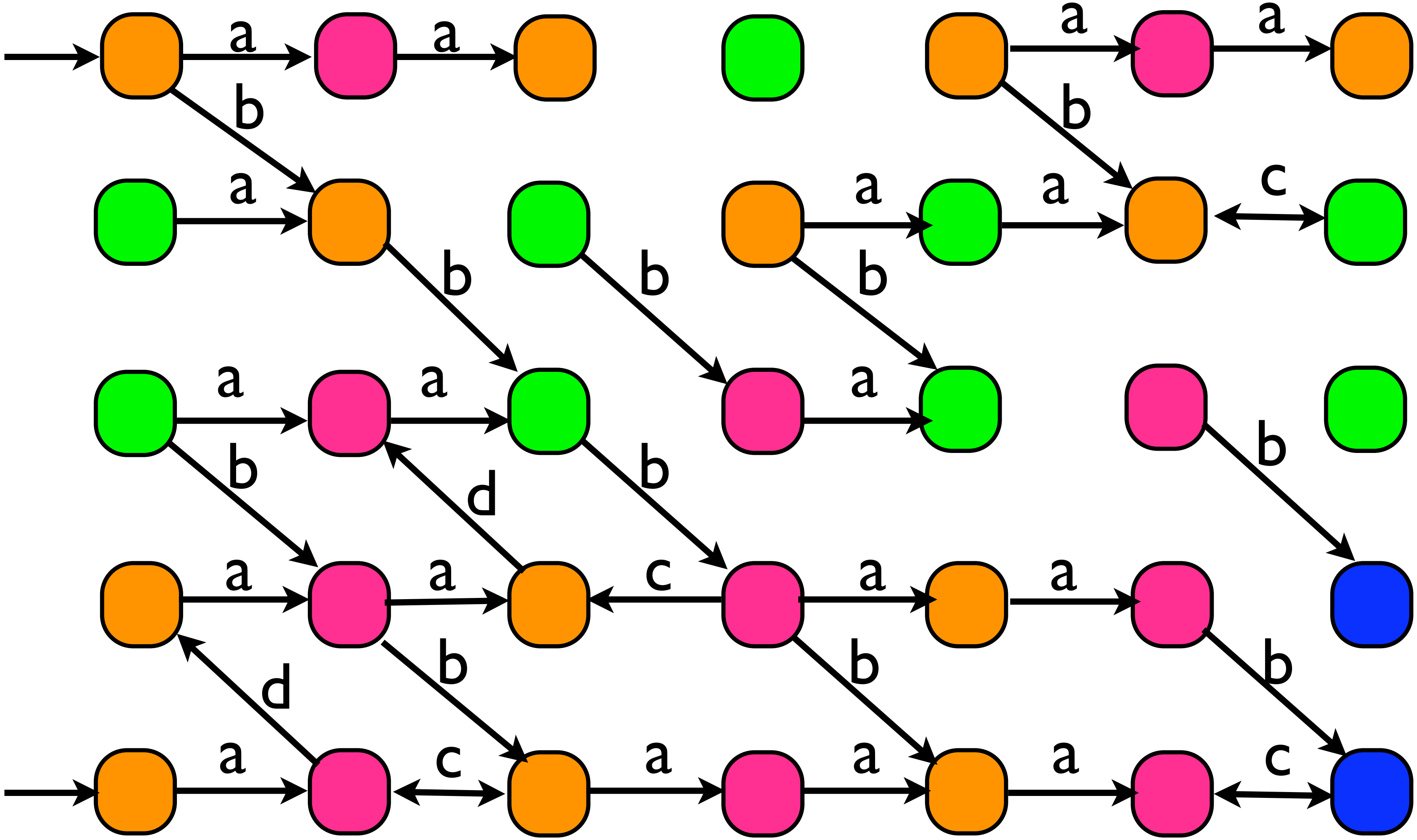
Labeled transition systems

- A **labeled transition system**, LTS for short, is a tuple $(S, S_0, \Sigma, T, C, \lambda)$ where:
  - S is a (finite or infinite) set of states
  - $S_0 \subseteq S$ is the subset of initial states
  - Σ is an event or action set (finite or infinite)
  - C is a (finite or infinite) set of colors
  - $\lambda : S \rightarrow C$ is a labeling function that labels each state with a color.

A labelled transition system:



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Reachability

- **Reachability verification problem**

  Instance: a LTS $(S,S_0,\Sigma,T,C,\lambda)$, a set $Goal \subseteq S$.
  Question: is there an execution of the LTS that starts in S_0 and reaches $Goal$?
  More formally, is there a sequence $s_0\sigma_0 s_1\sigma_1 s_2\sigma_2 \ldots \sigma_{n-1}s_n$ such that
  (1) $s_0 \in S_0$, (2) $\forall i \cdot 0 \leq i < n \cdot T(s_i,\sigma_i,s_{i+1})$, and (3) $s_n \in Goal$?

- The **set of reachable states** of a LTS $(S,S_0,\Sigma,T,C,\lambda)$ is the set of states $s \in S$ such that
  there is a sequence $s_0\sigma_0 s_1\sigma_1 s_2\sigma_2 \ldots \sigma_{n-1}s_n$
  and (1) $s_0 \in S_0$, (2) $\forall i \cdot 0 \leq i < n \cdot T(s_i,\sigma_i,s_{i+1})$, (3) $s_n = s$.

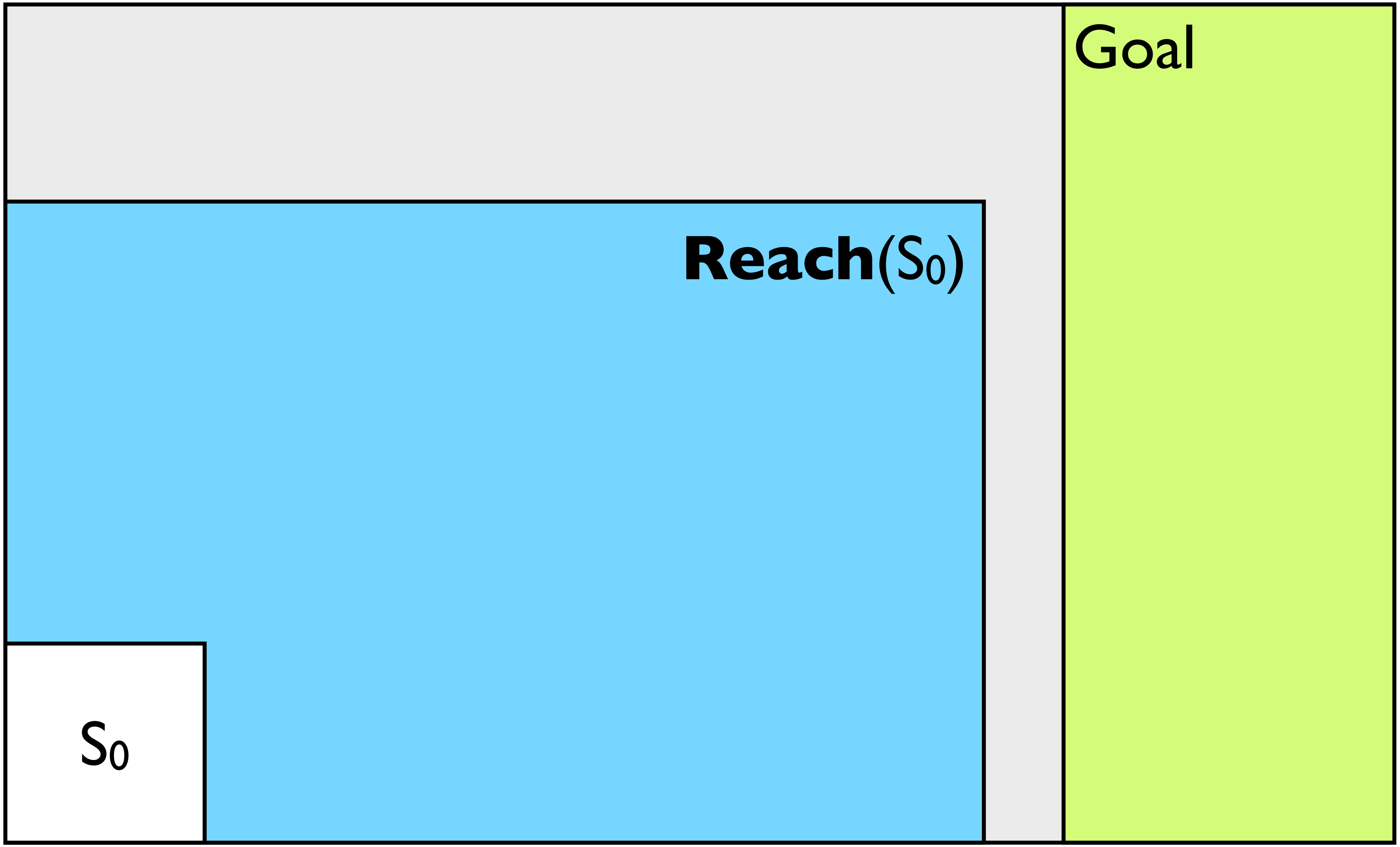
  Let **Reach**(S_0) denote the set of reachable states.

- Clearly, there is a path that starts in S_0 and reaches G <u>iff</u> $\mathbf{Reach(S_0) \cap Goal \neq \emptyset}$.

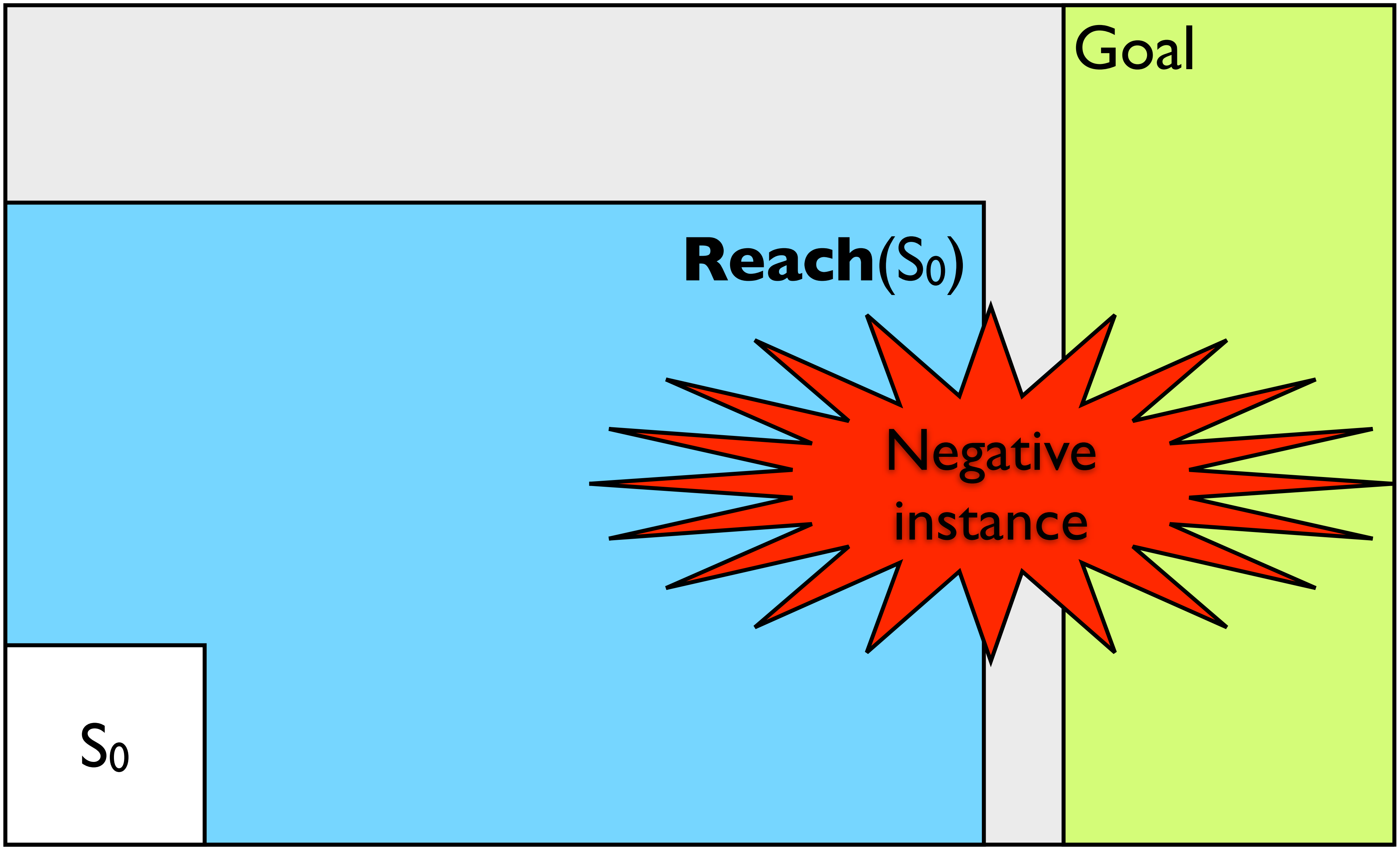
Reachability

Reachability

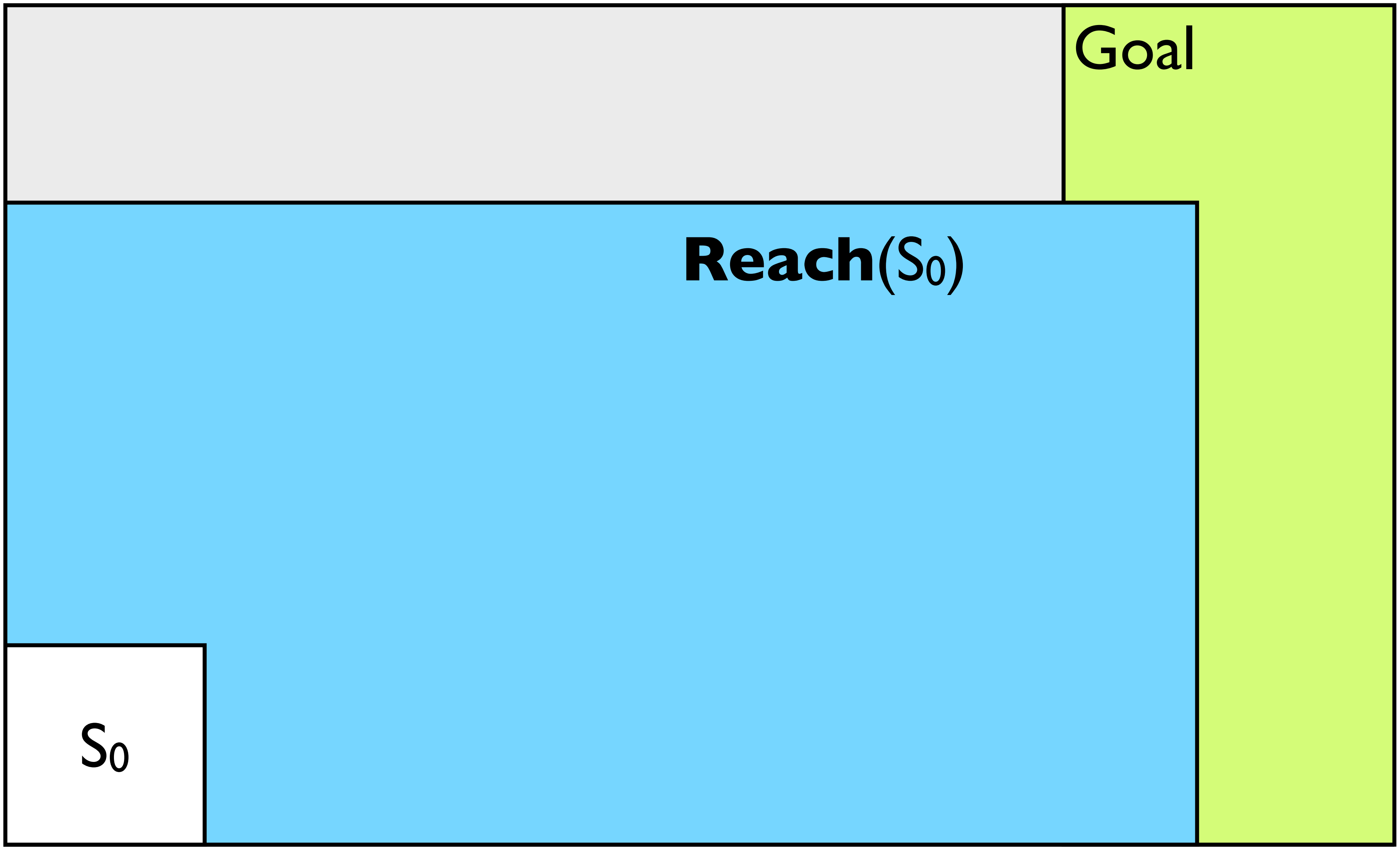
Reachability



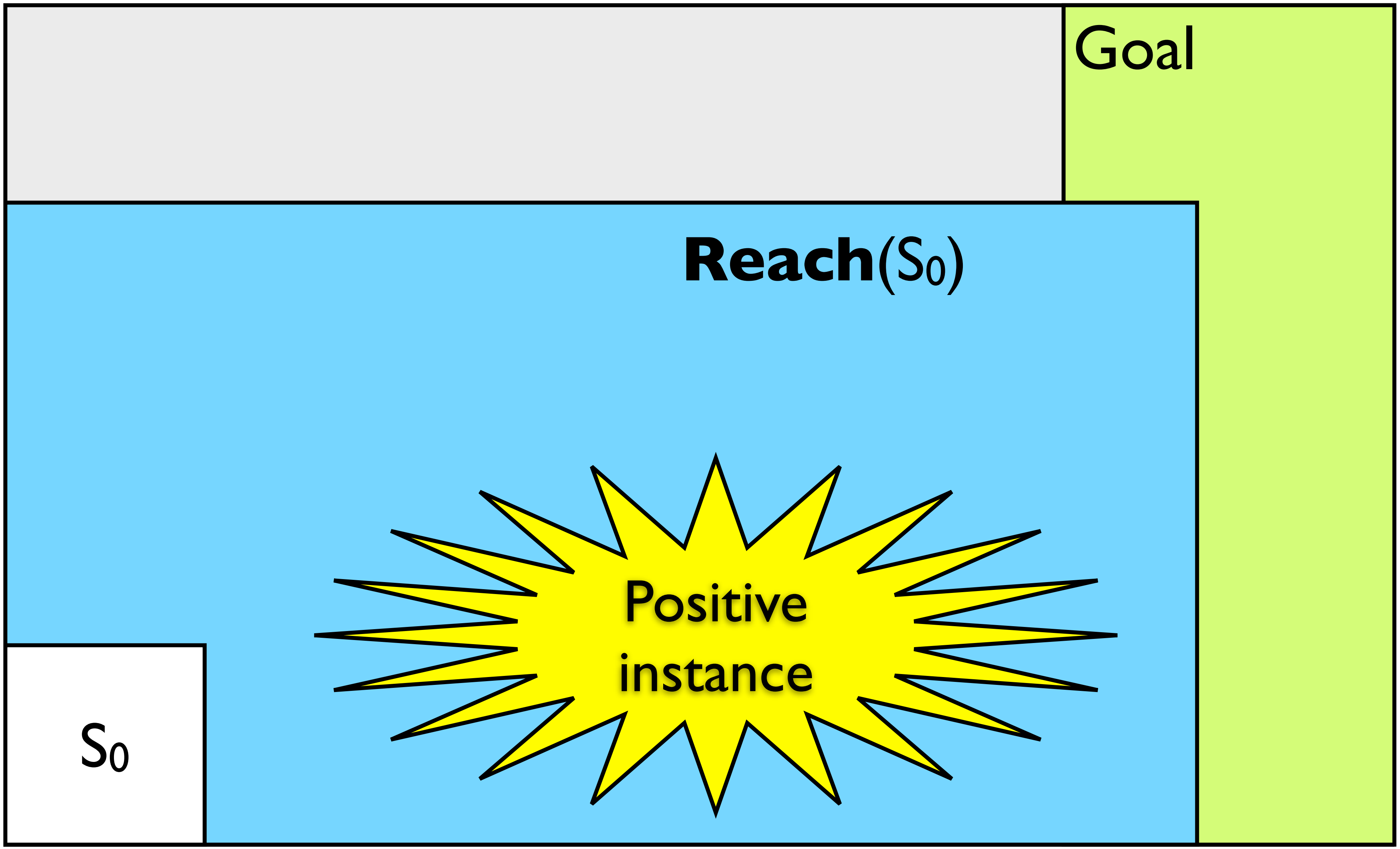
Reachability



Reachability



Reachability



Safety

- **Safety verification problem**

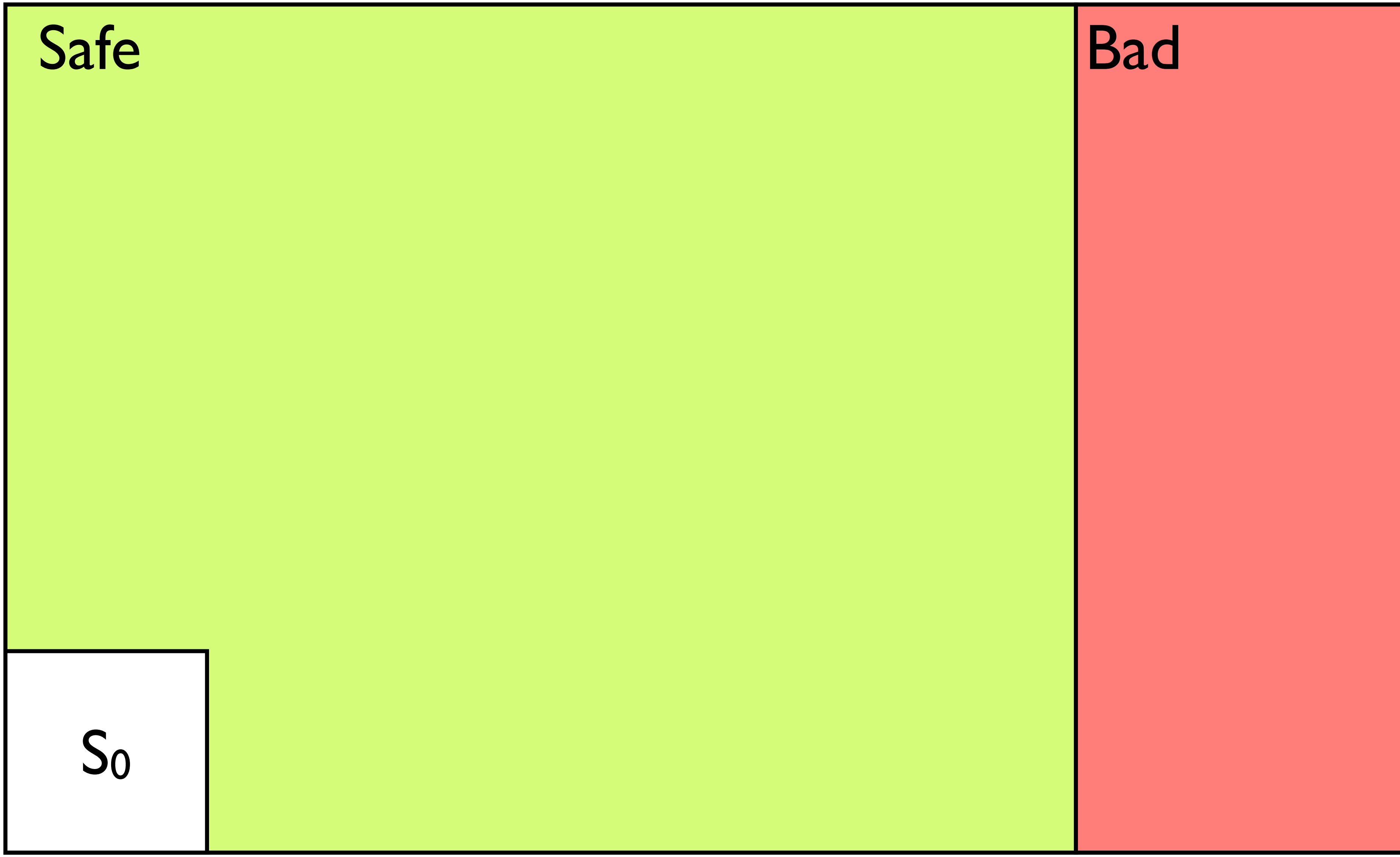
  Instance: a LTS $(S,S_0,\Sigma,T,C,\lambda)$, a set of states $Safe \subseteq S$.
  Question: are all paths that starts in S_0 staying within $Safe$.
  More formally, for all sequences $s_0\sigma_0s_1\sigma_1s_2\sigma_2\ldots\sigma_{n-1}s_n$ such that
  (1) $s_0 \in S_0$, (2) $\forall i \cdot 0 \leq i < n \cdot T(s_i,\sigma_i,s_{i+1})$,
  is it the case that (3) $\forall i \cdot 0 \leq i \leq n \cdot s_i \in Safe$?

- Clearly all paths that start in S_0 are staying within Safe
  iff **Reach(S_0)**$\cap$**(S\Safe)=**$\varnothing$.

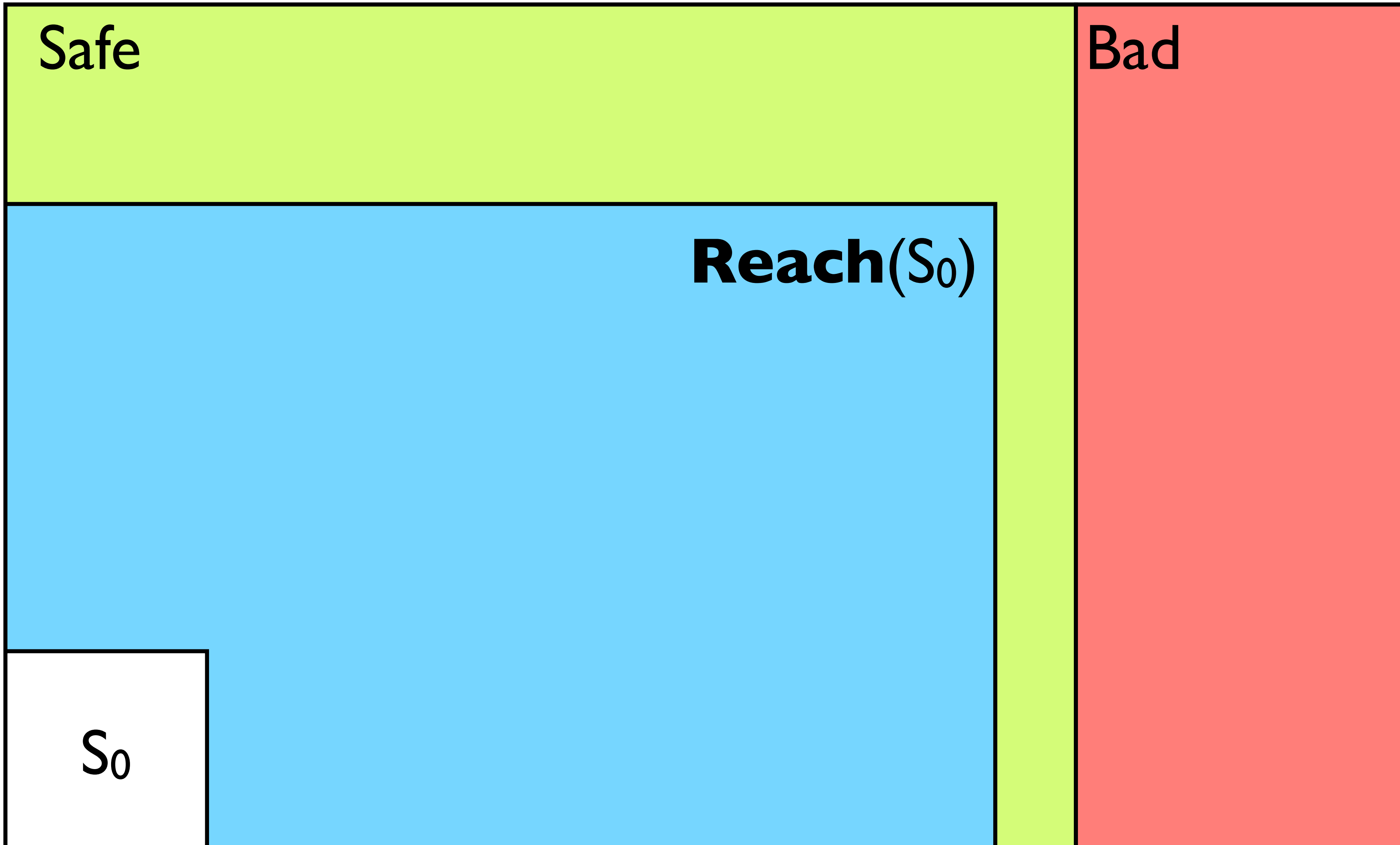
- So, the safety and reachability problems are **dual problems**.

Safety

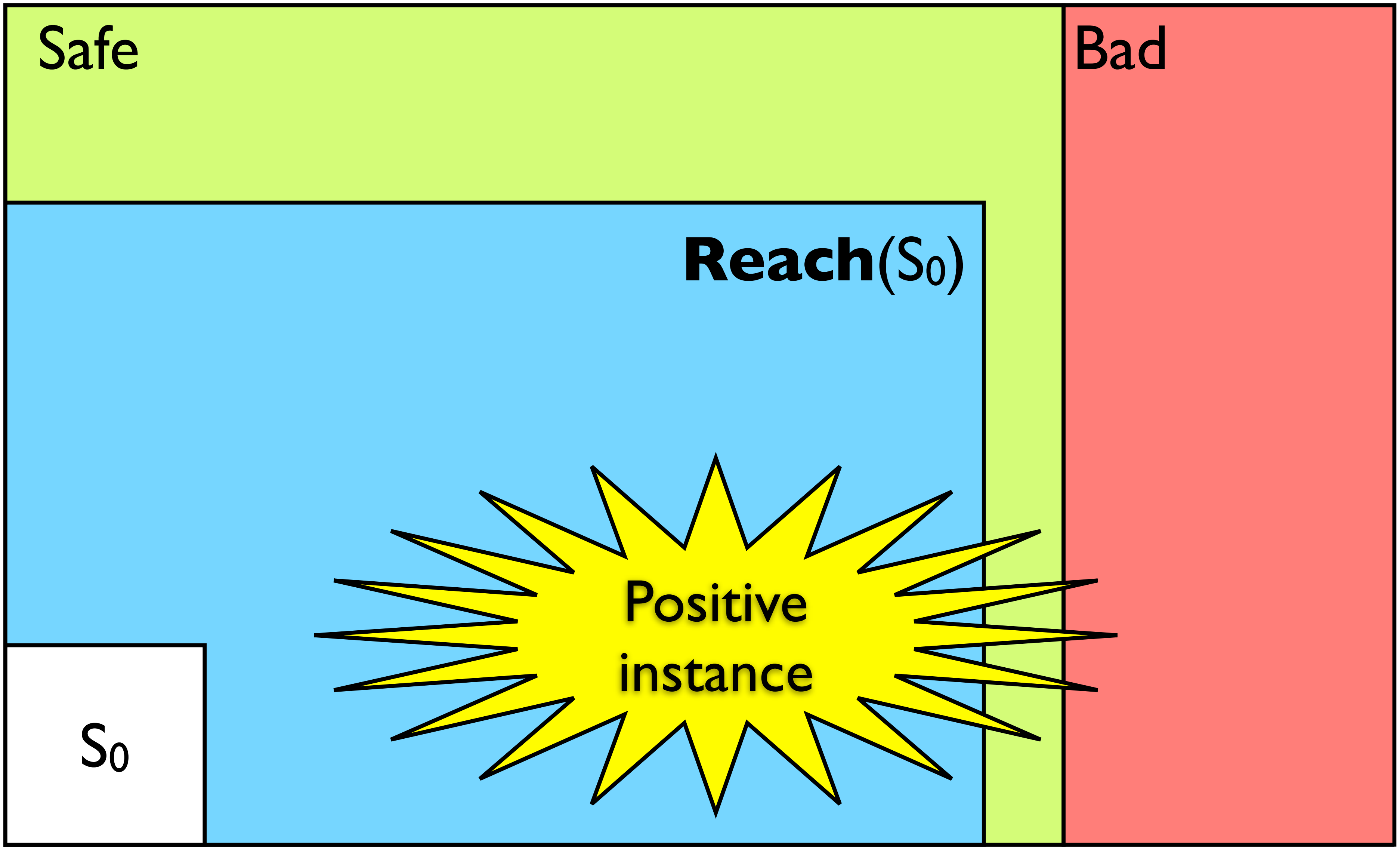
Safety



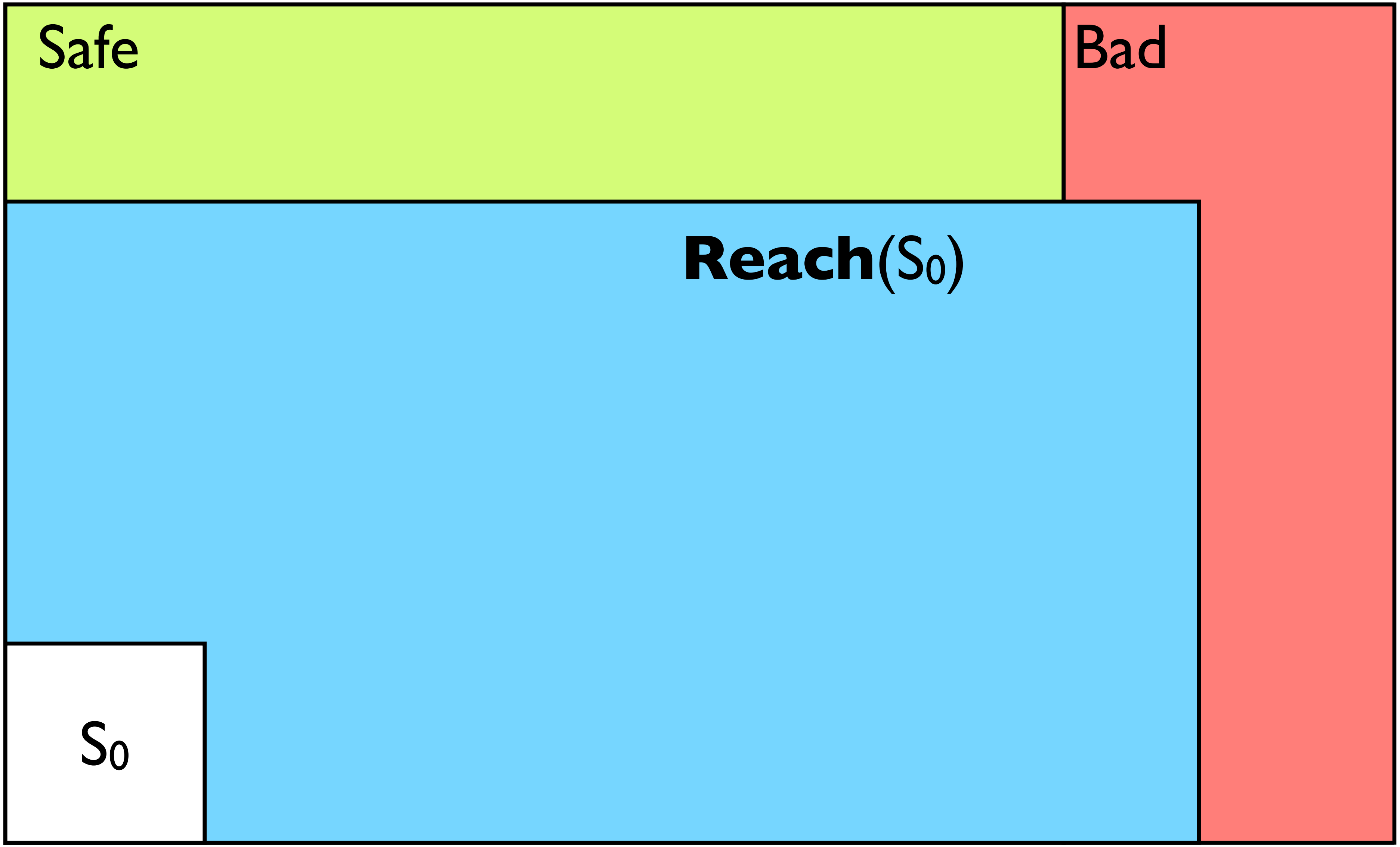
Safety



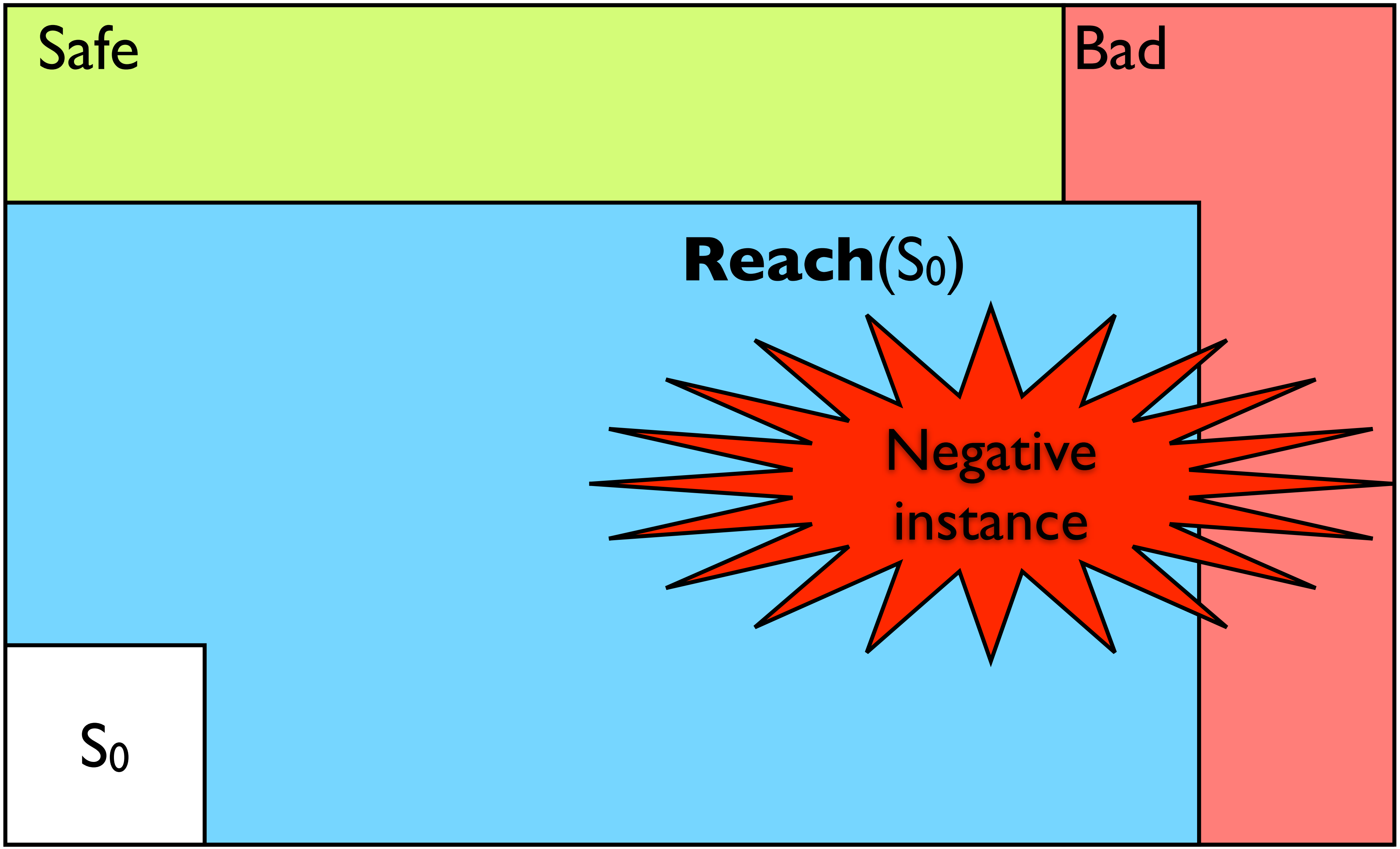
Safety



Safety



Safety

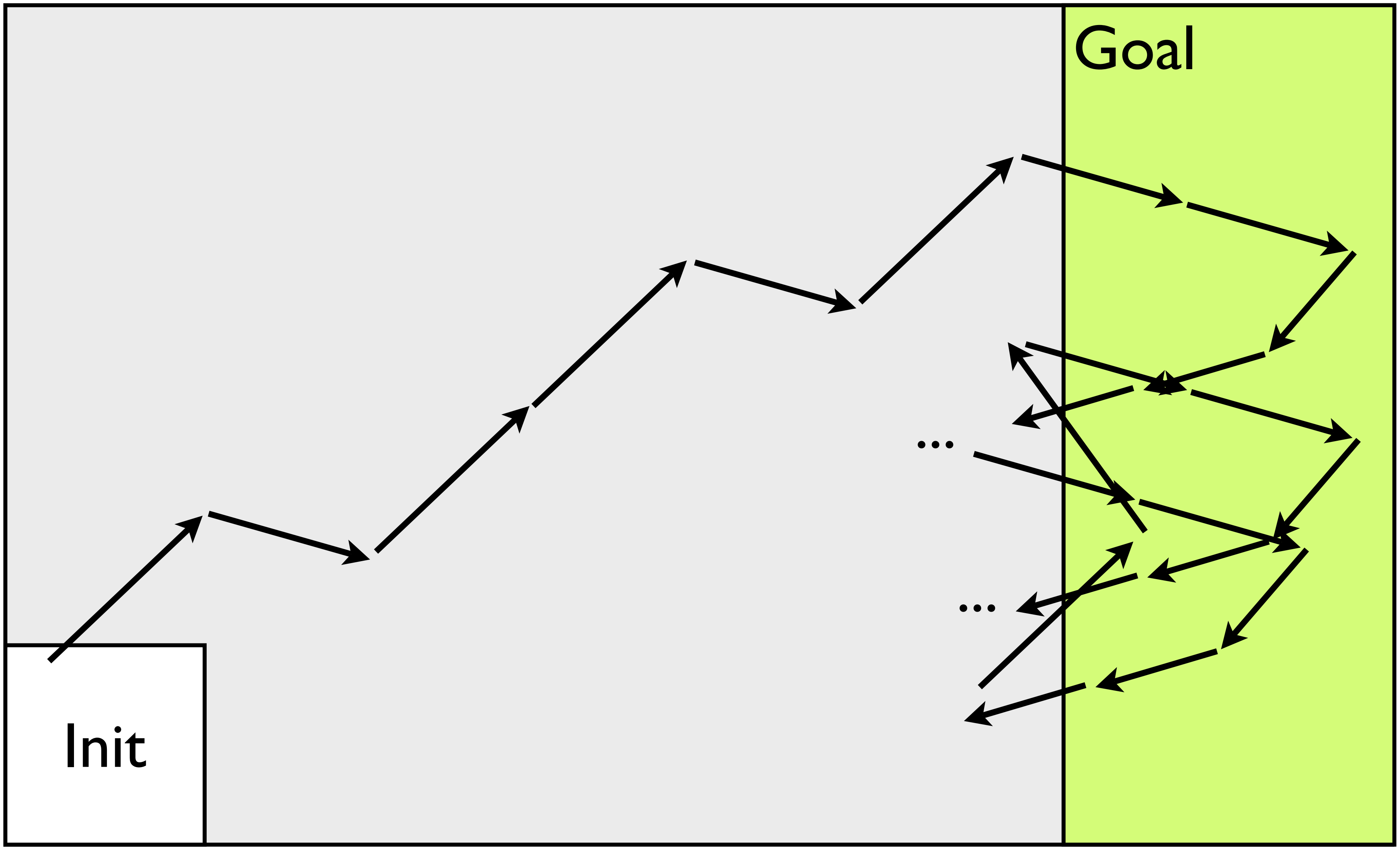


Büchi condition

- **Büchi verification problem**

  Instance: a LTS $(S,S_0,\Sigma,T,C,\lambda)$, a set Goal $\subseteq$ S.
  Question: is there one execution of the LTS that starts in S_0 and passes infinitely often by the set Goal $\subseteq$ S ?
  More formally, is there an execution $s_0\sigma_0 s_1\sigma_1 s_2\sigma_2 \ldots \sigma_{n-1} s_n \ldots$ such that
  (1) $s_0 \in S_0$,
  (2) $\forall i \cdot 0 \leq i \cdot T(s_i,\sigma_i,s_{i+1})$,
  (3) $\forall i \geq 0 \cdot \exists j \geq i$ such that $s_j \in$ Goal ?

Büchi condition



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Post, Pre and Apre operators

- We will design verification algorithms for the reachability, safety and Büchi properties.
- Our algorithms will manipulate **sets of states**.
- Besides set operations, we will need to compute the set of states that are successors (Post), or predecessors (Pre and Apre) of a set of states.

The **Post** : $2^S \times \Sigma \rightarrow 2^S$
takes (1) a set of states X
(2) an action σ
and returns the set of successors of X by σ

Post$(X,\sigma)=\{\ y \in X \mid \exists\ x \in X\ \cdot T(x,\sigma,y)\ \}$

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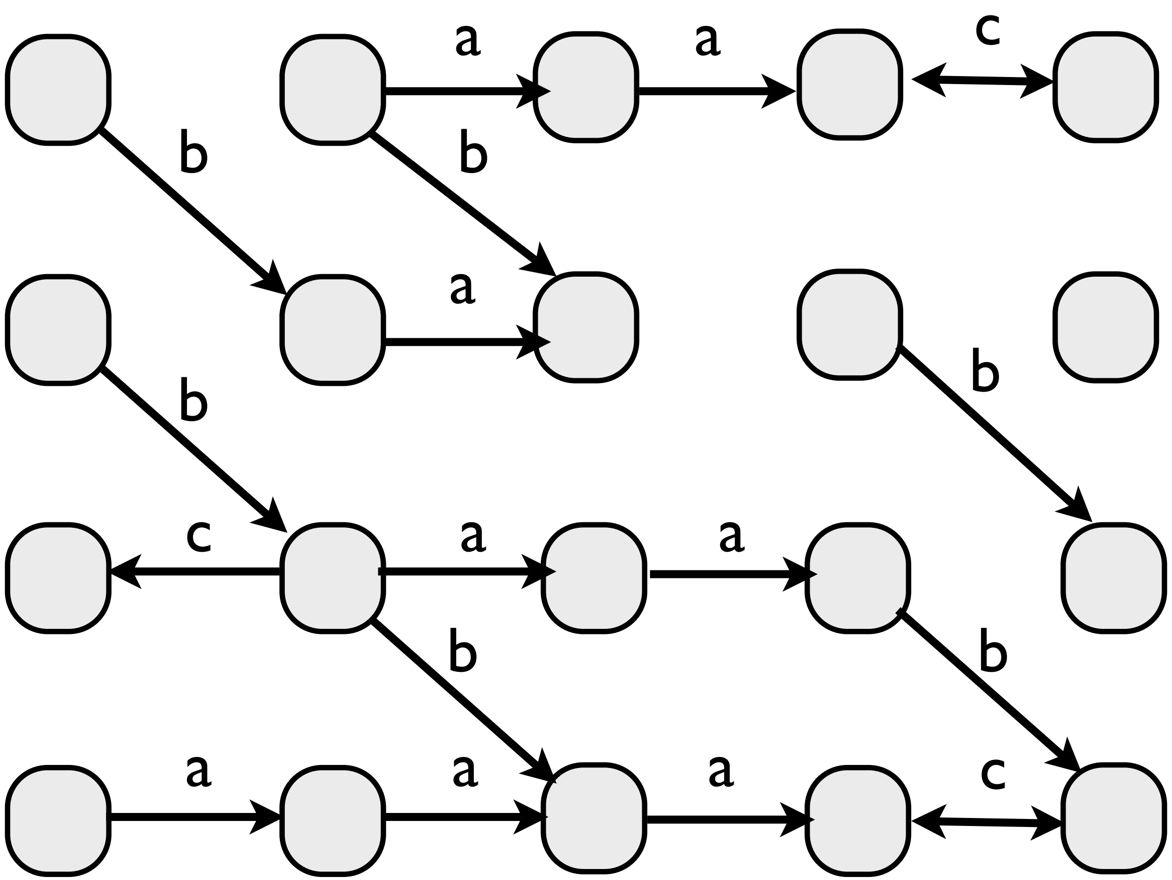

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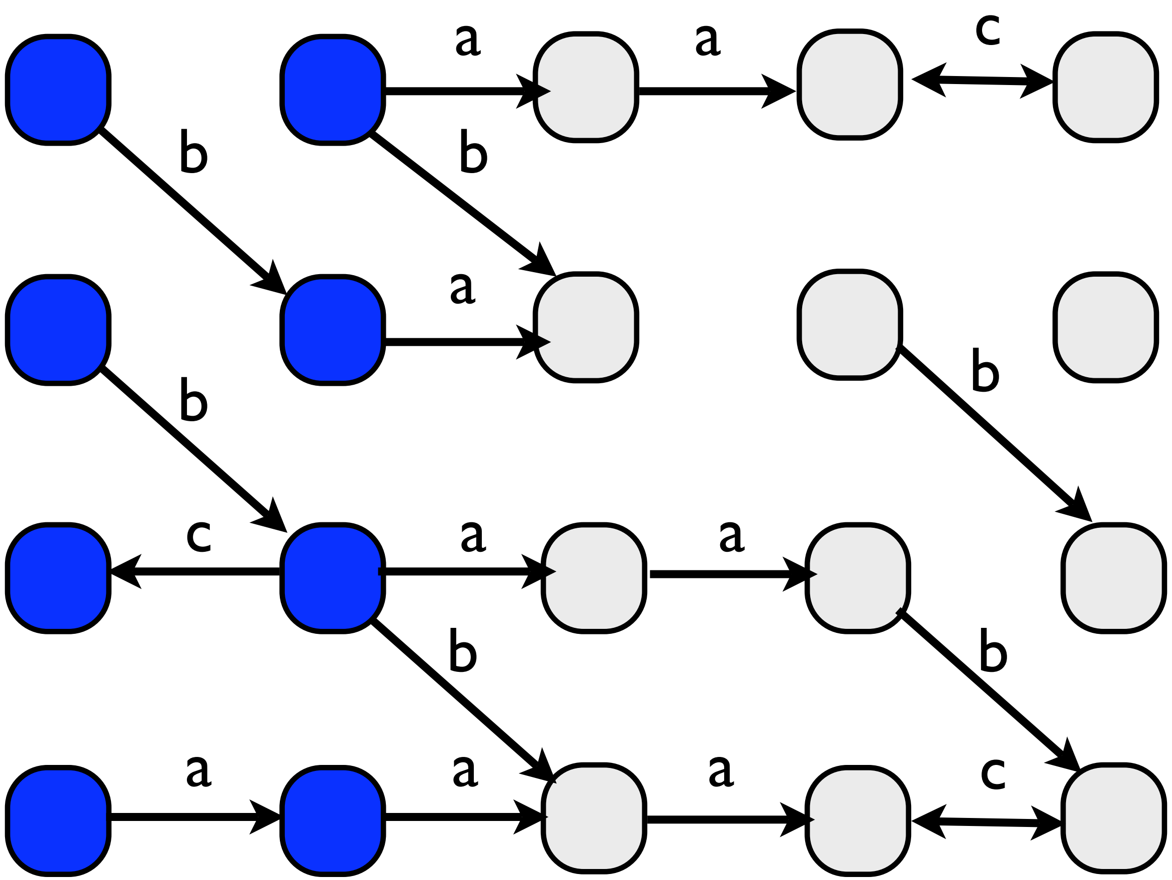
Post(X,b)=



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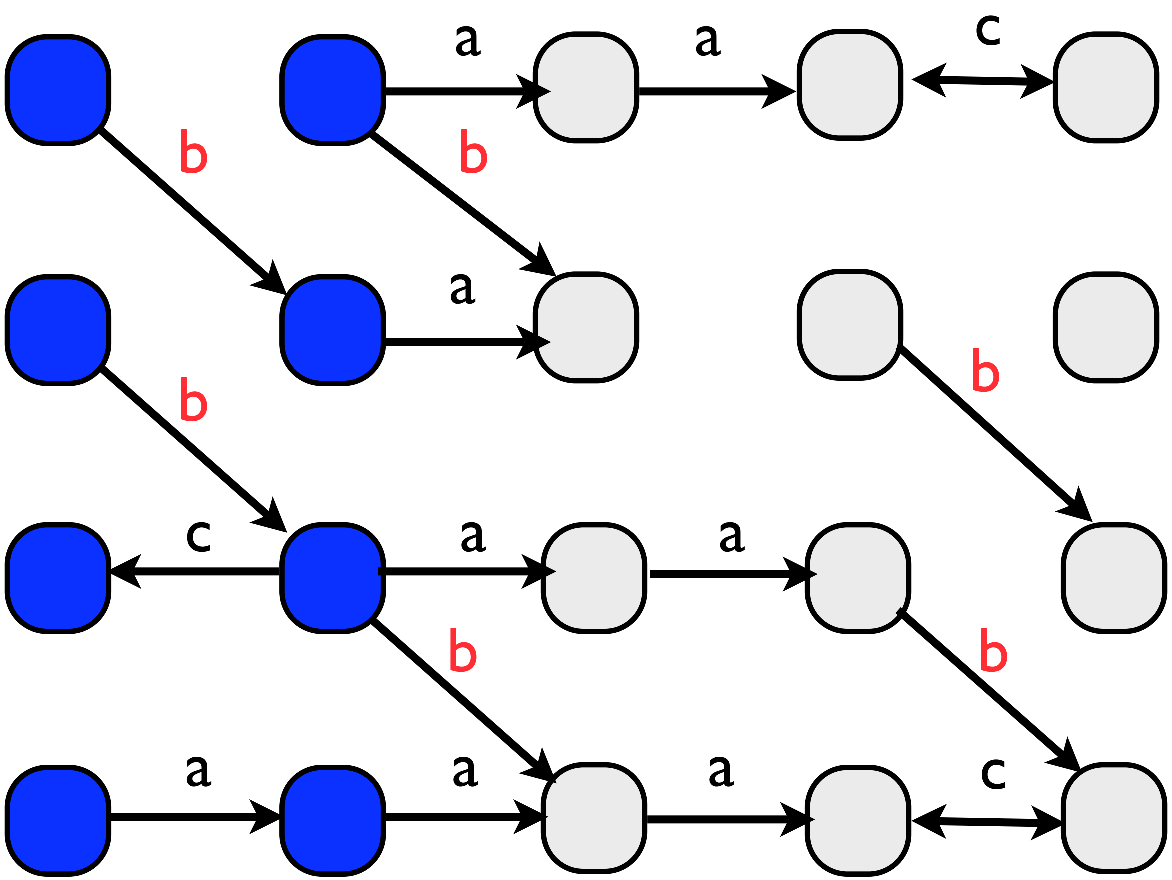

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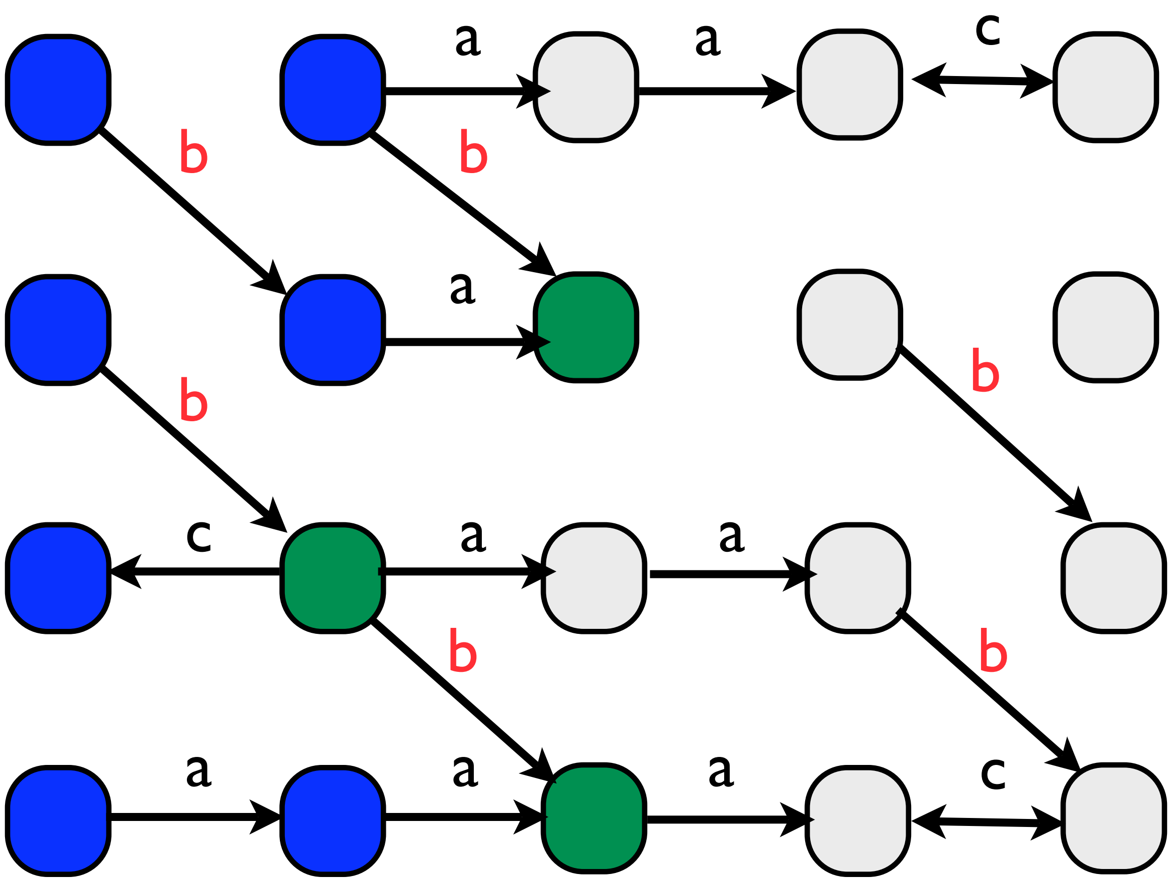
Post(X,b)=Y



The **Pre** : $2^S \times \Sigma \rightarrow 2^S$

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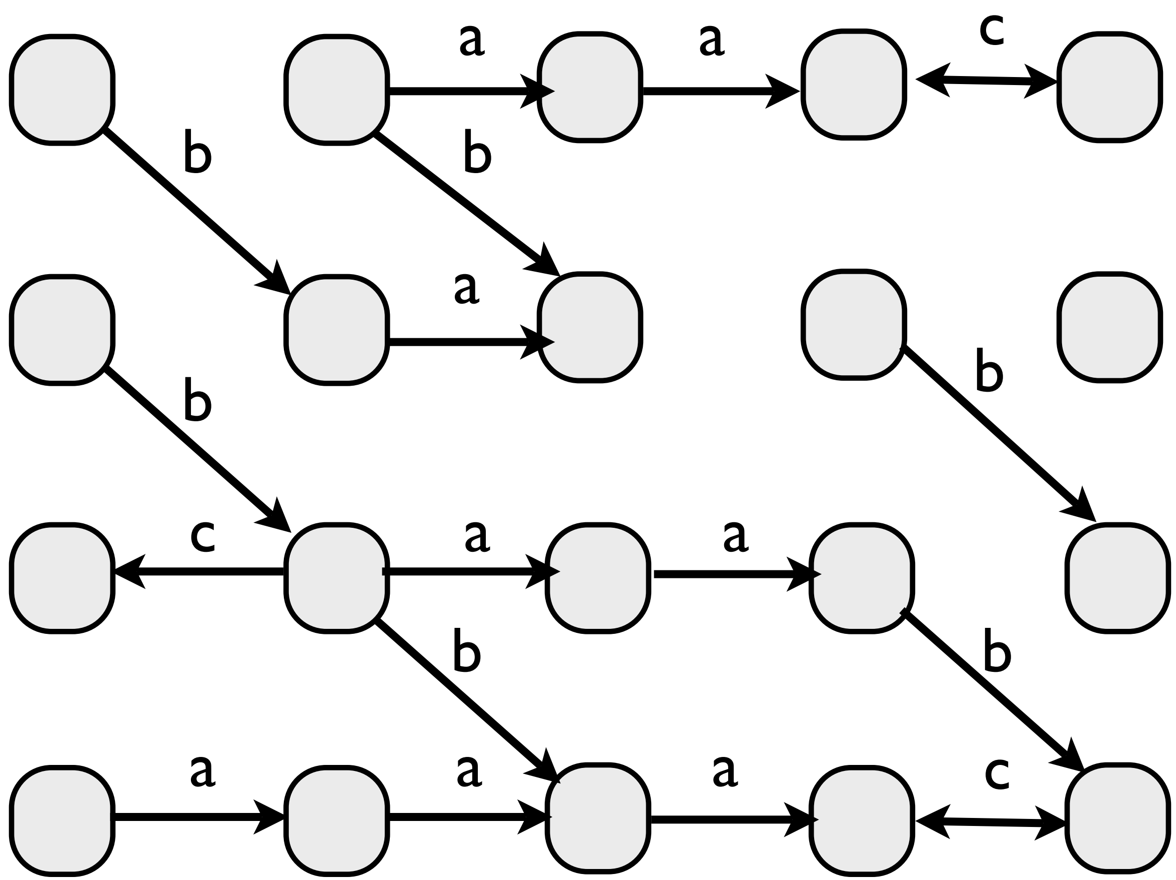

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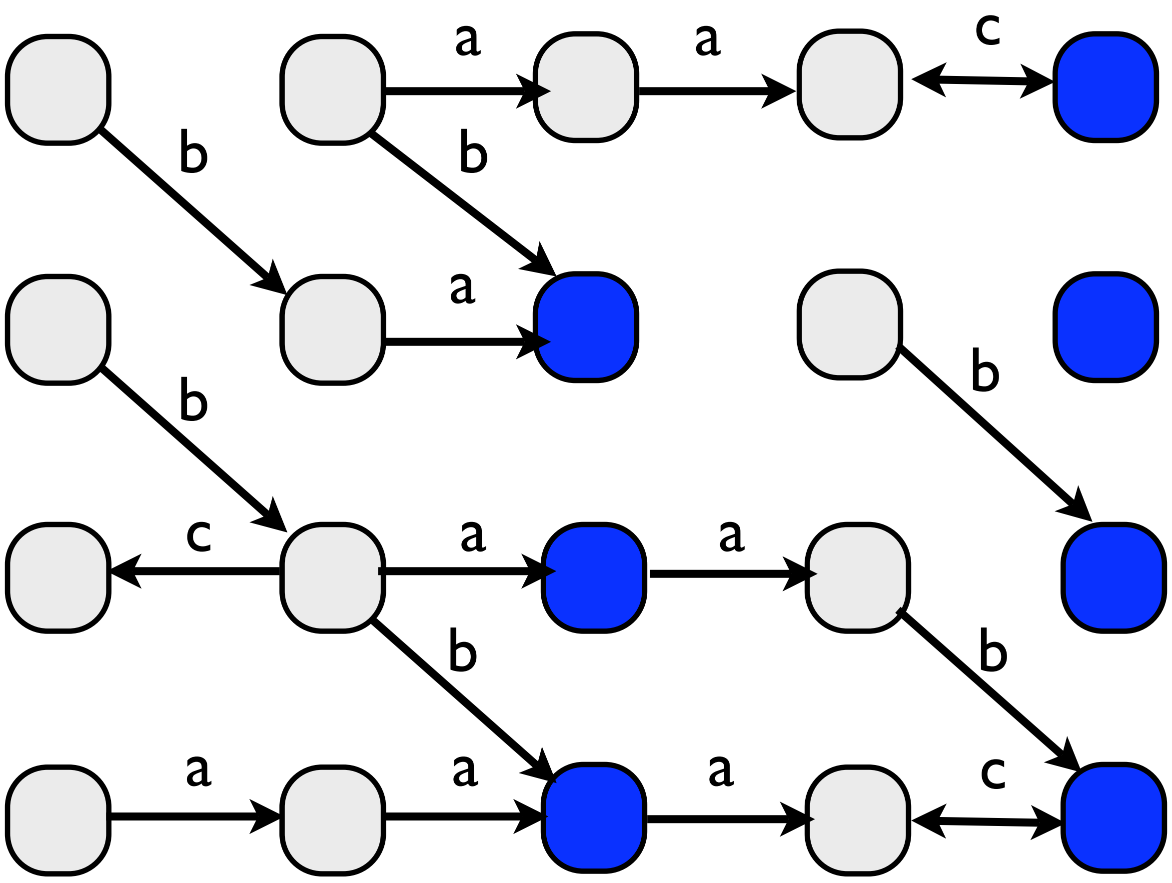

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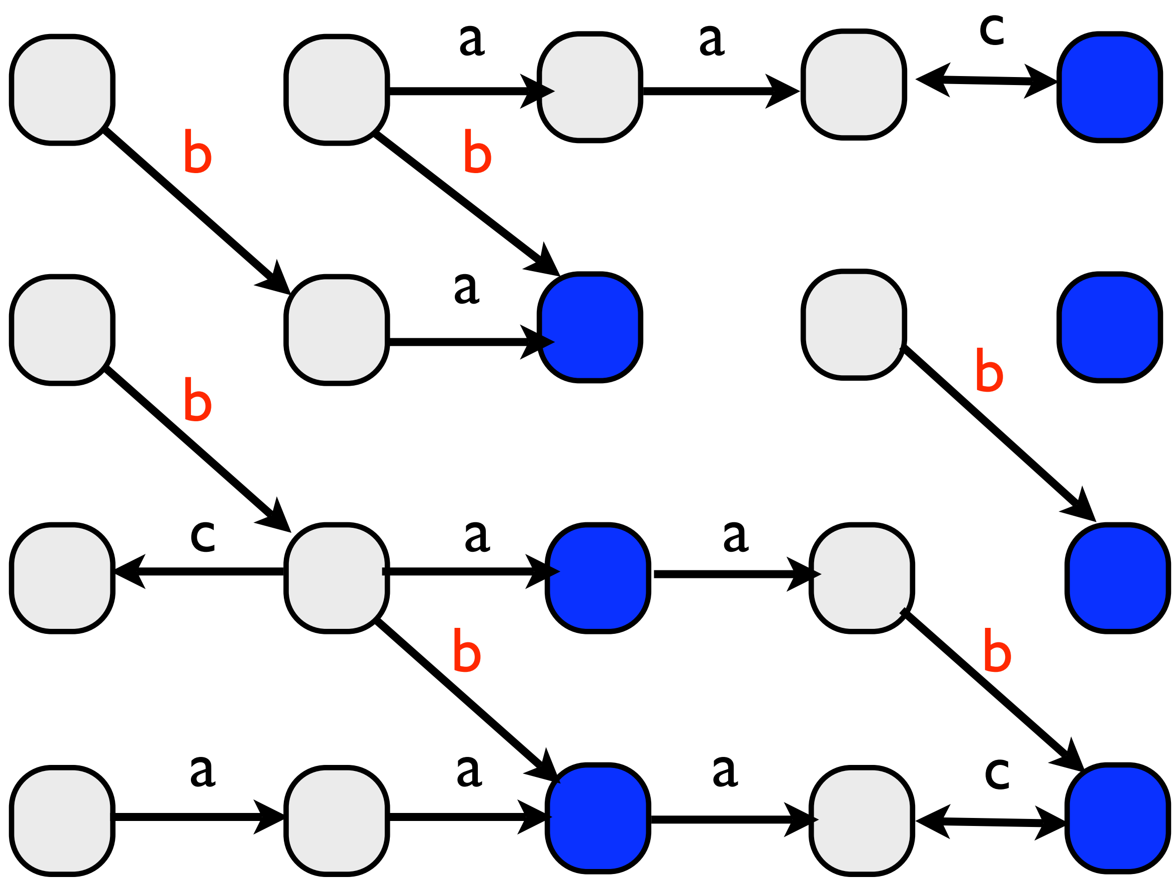

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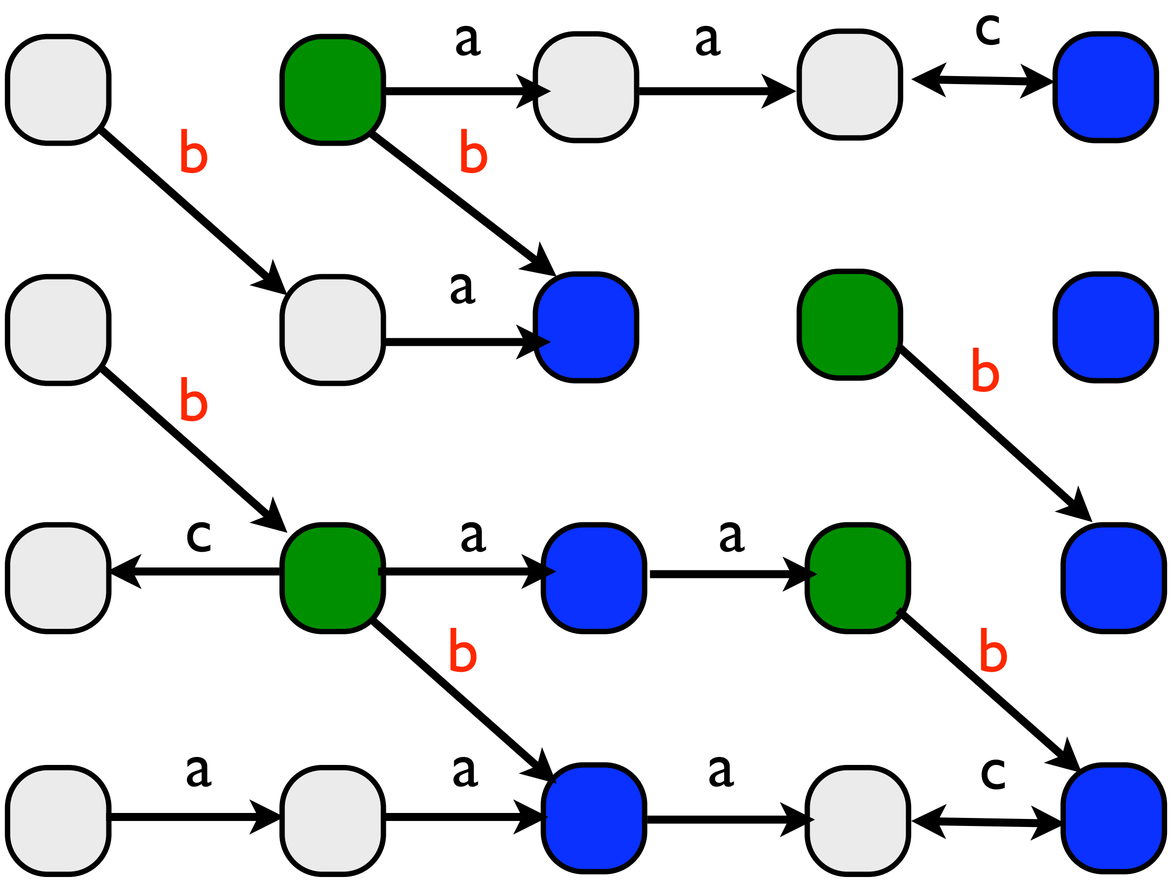
$\mathbf{Pre}(X,b)=Y$



The **Apre** : $2^S \times \Sigma \rightarrow 2^S$

takes (1) a set of states X

(2) an action σ

and returns the set of states that have all their successors by σ in X

Apre$(X,\sigma)=\{ x \in S \mid \forall\, y \in S \cdot T(x,\sigma,y) \Rightarrow y \in X \}$

The **Apre** : $2^S \times \Sigma \rightarrow 2^S$

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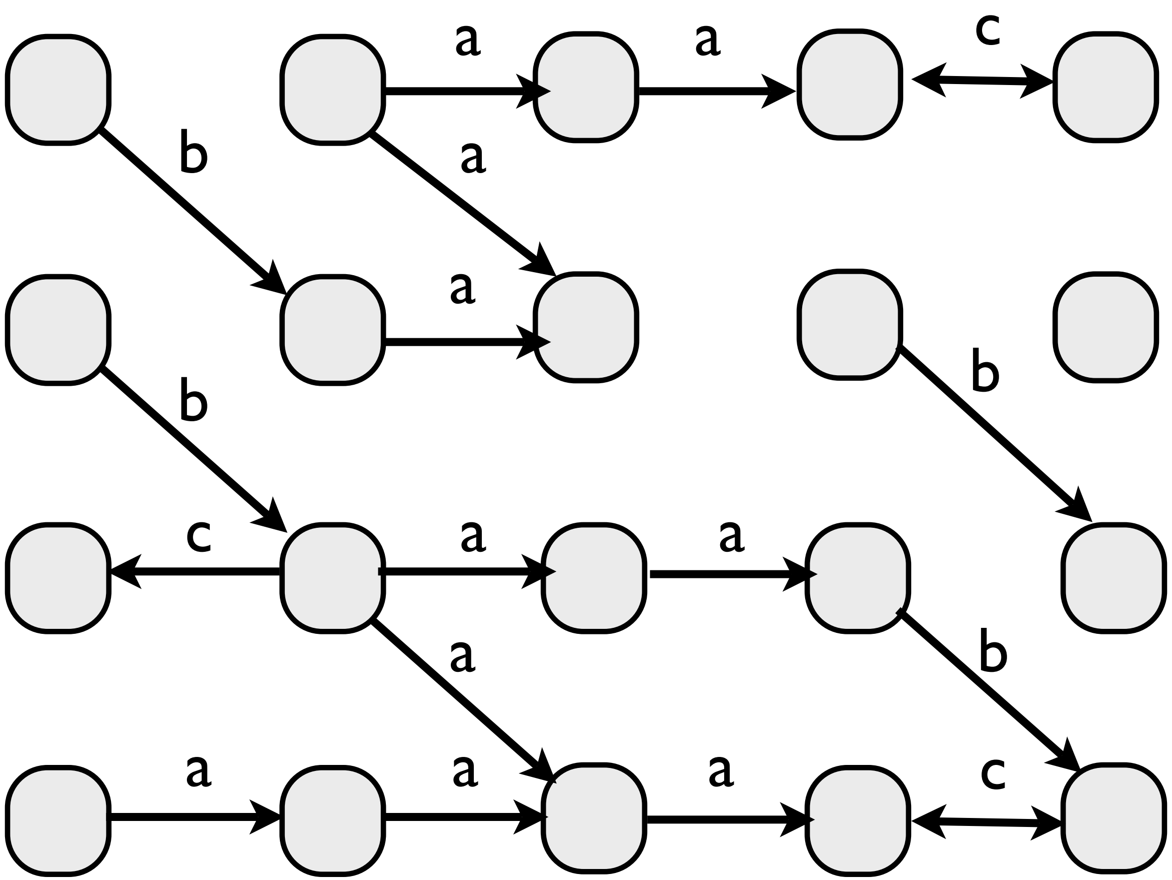

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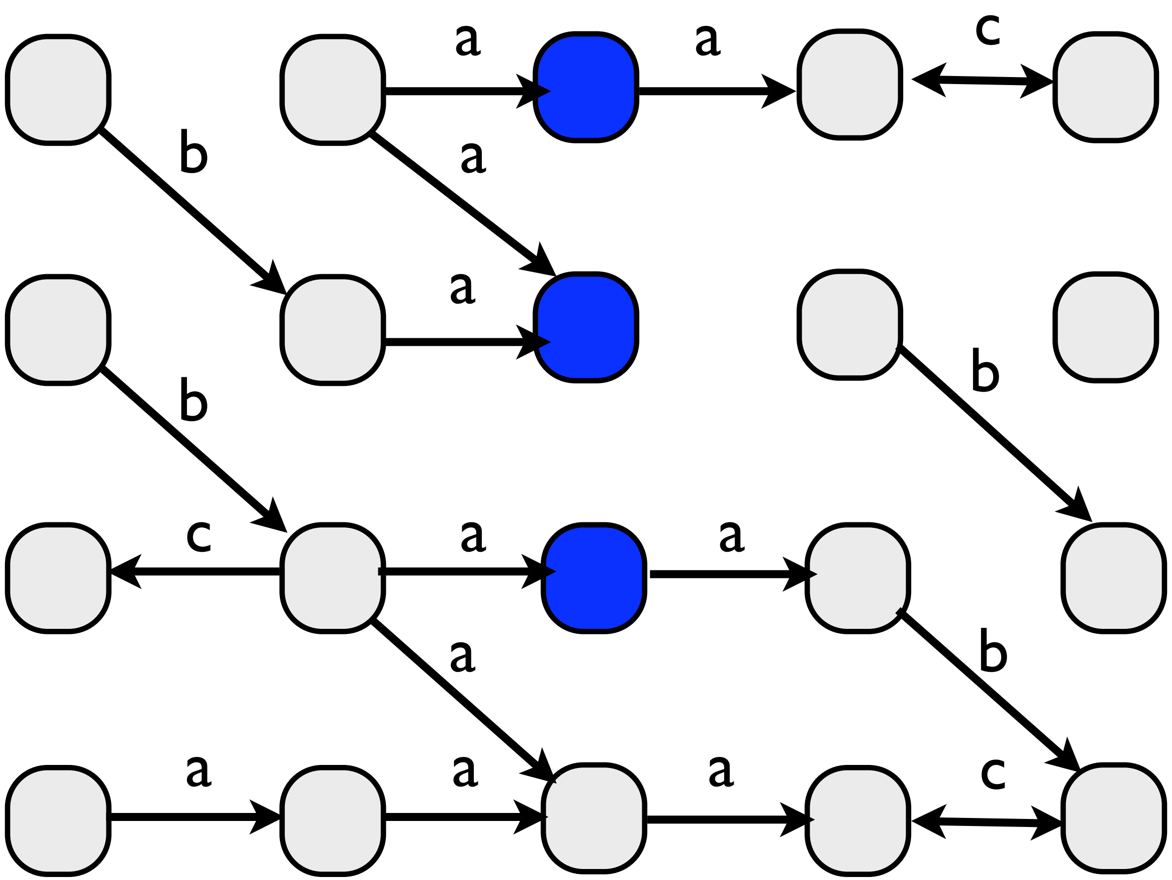
Apre(X,a)



The **Apre** : $2^S \times \Sigma \rightarrow 2^S$
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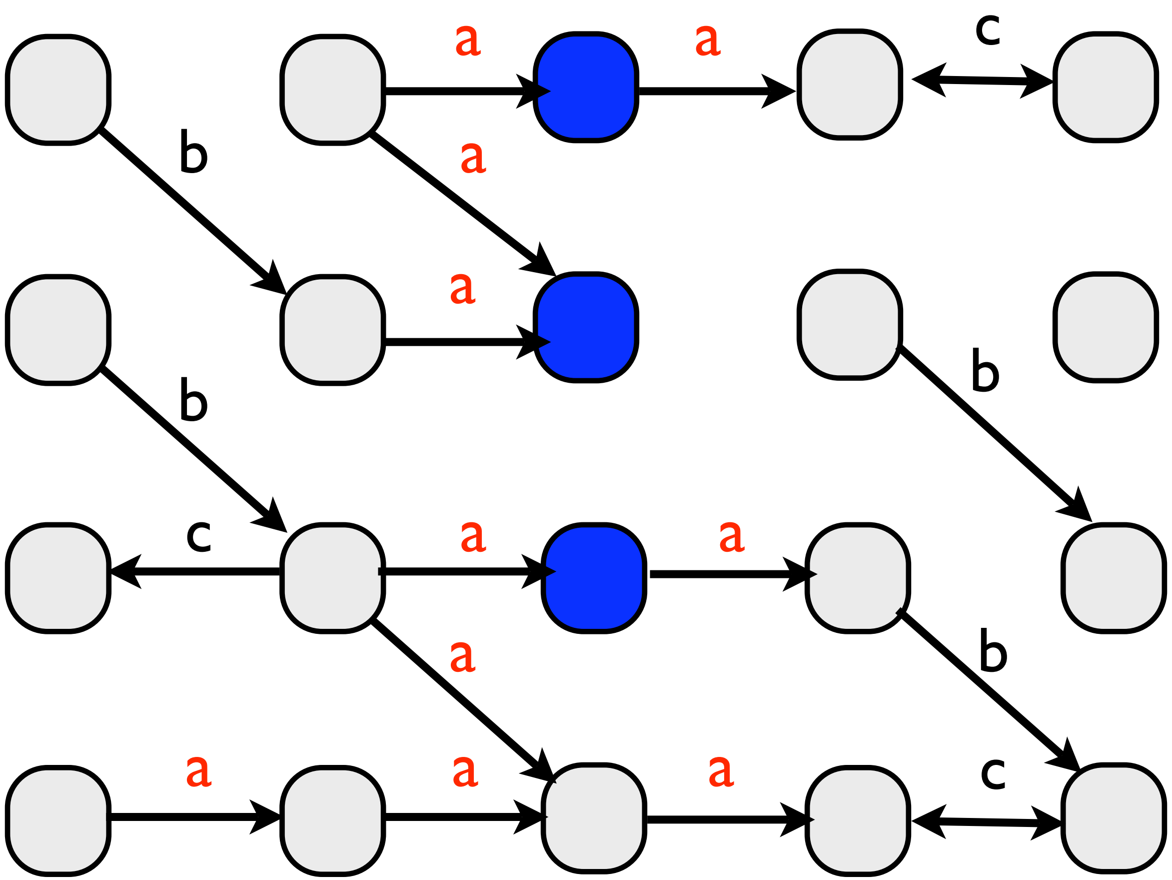
Apre(X,a)



The **Apre** : $2^S \times \Sigma \rightarrow 2^S$
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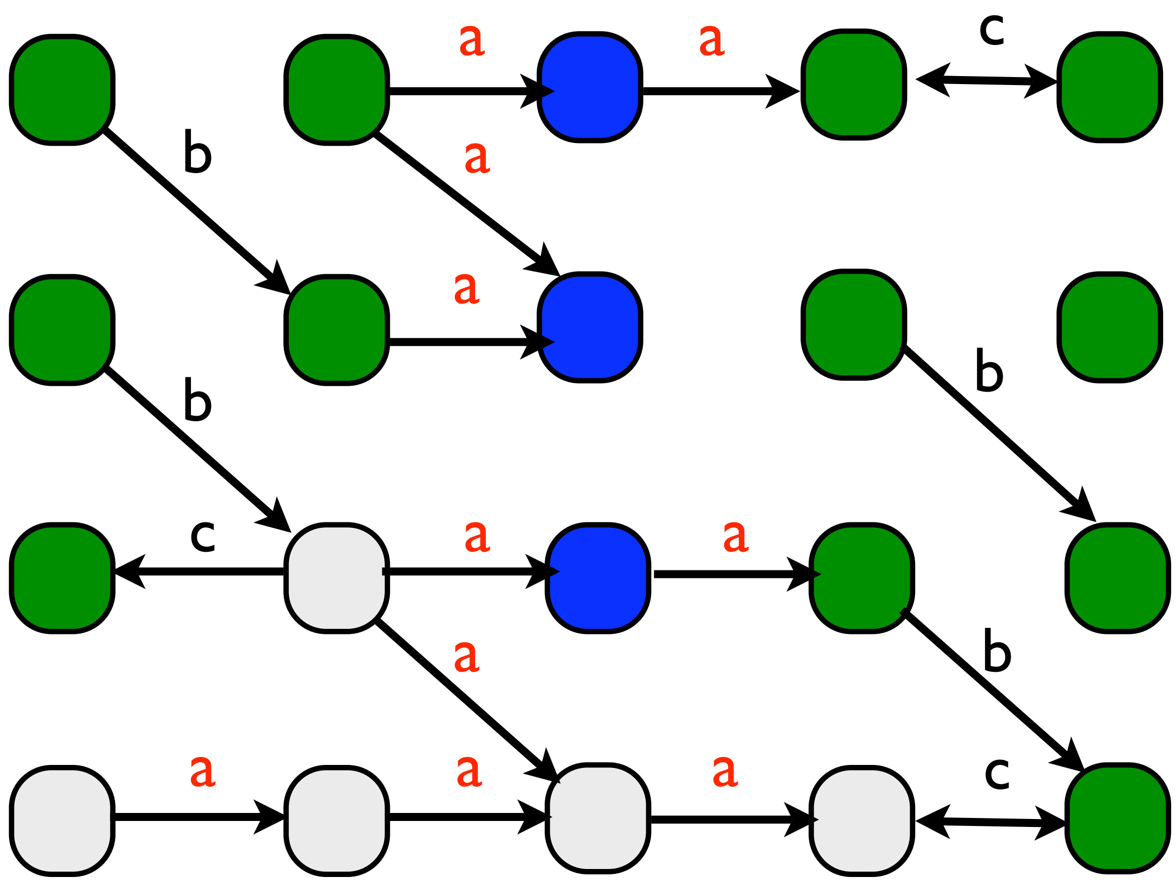
$$\mathbf{Apre}(X,\sigma)=\{\ x \in S \mid \forall\ y \in S \cdot T(x,\sigma,y) \Rightarrow y \in X\ \}$$

Apre(X,a)=Y



PRE-POST-APRE

From the **Pre** : $2^S \times \Sigma \rightarrow 2^S$, the **Post** : $2^S \times \Sigma \rightarrow 2^S$, and the **Apre** : $2^S \times \Sigma \rightarrow 2^S$, we can define their generalizations over the entire alphabet of actions:

The **POST** : $2^S \rightarrow 2^S$ takes a set of states X and returns the set of states Y that are reachable in one step from X, i.e. :

POST$(X)=\{ y \in S \mid \exists x \in X \cdot \exists \sigma \in \Sigma \cdot T(x,\sigma,y) \}$

The **PRE** : $2^S \rightarrow 2^S$ takes a set of states X and returns the set of states Y that can reach X in one step, i.e. :

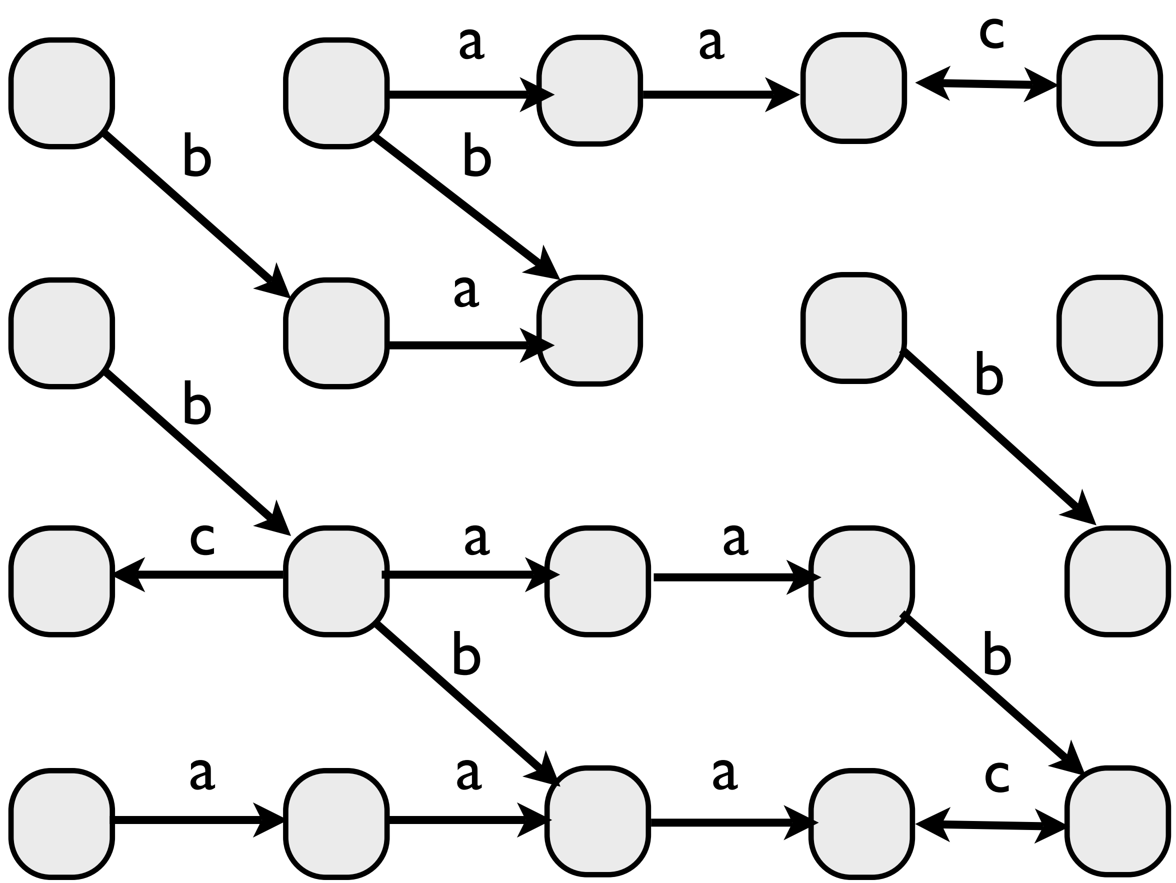
PRE$(X)=\{ y \in S \mid \exists x \in X \cdot \exists \sigma \in \Sigma \cdot T(y,\sigma,x) \}$

The **APRE**: $2^S \rightarrow 2^S$ takes a set of states X and returns the set of states Y that have all their one step successors in X, i.e. :

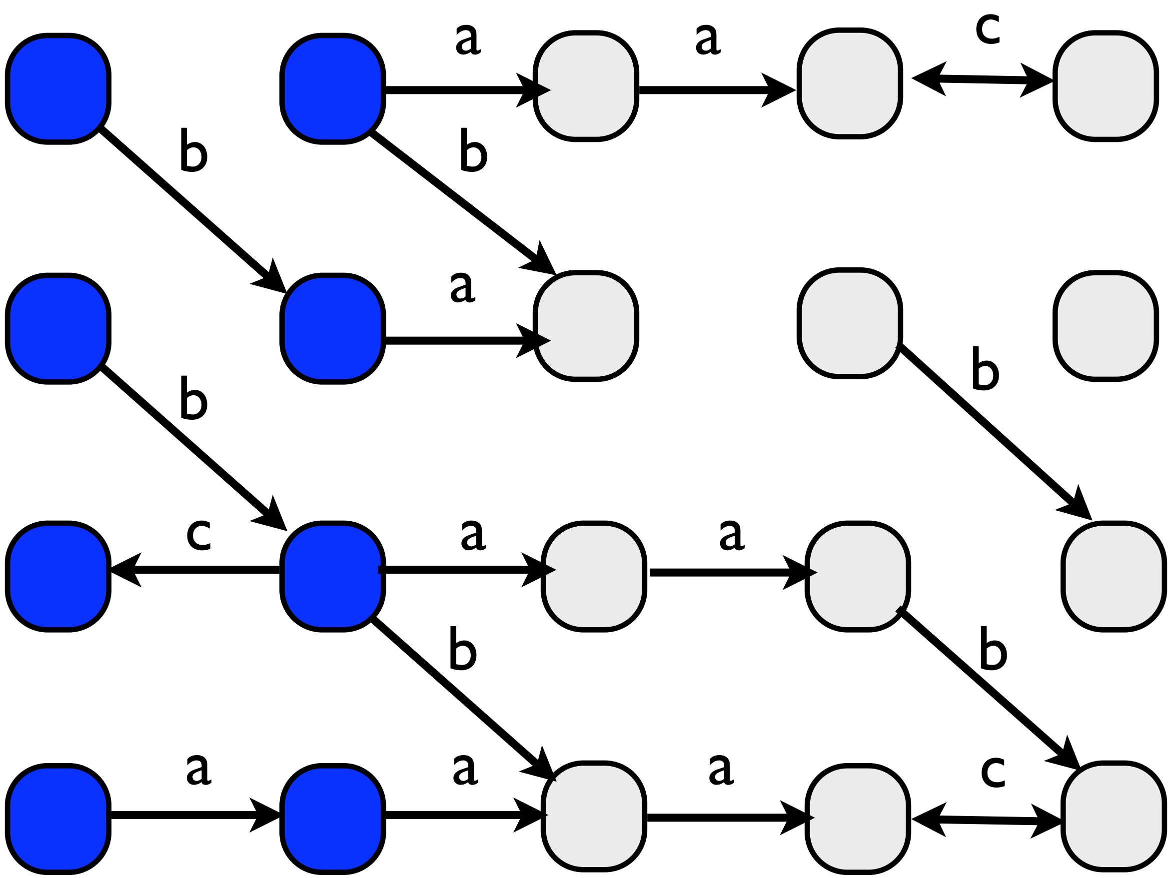
APRE$(X)=\{ y \in S \mid \forall x \in S \cdot \forall \sigma \in \Sigma \cdot T(y,\sigma,x) \Rightarrow x \in X \}$

Exercise : proof that **APRE**$(X)=S \setminus$**PRE**$(S \setminus X)$.

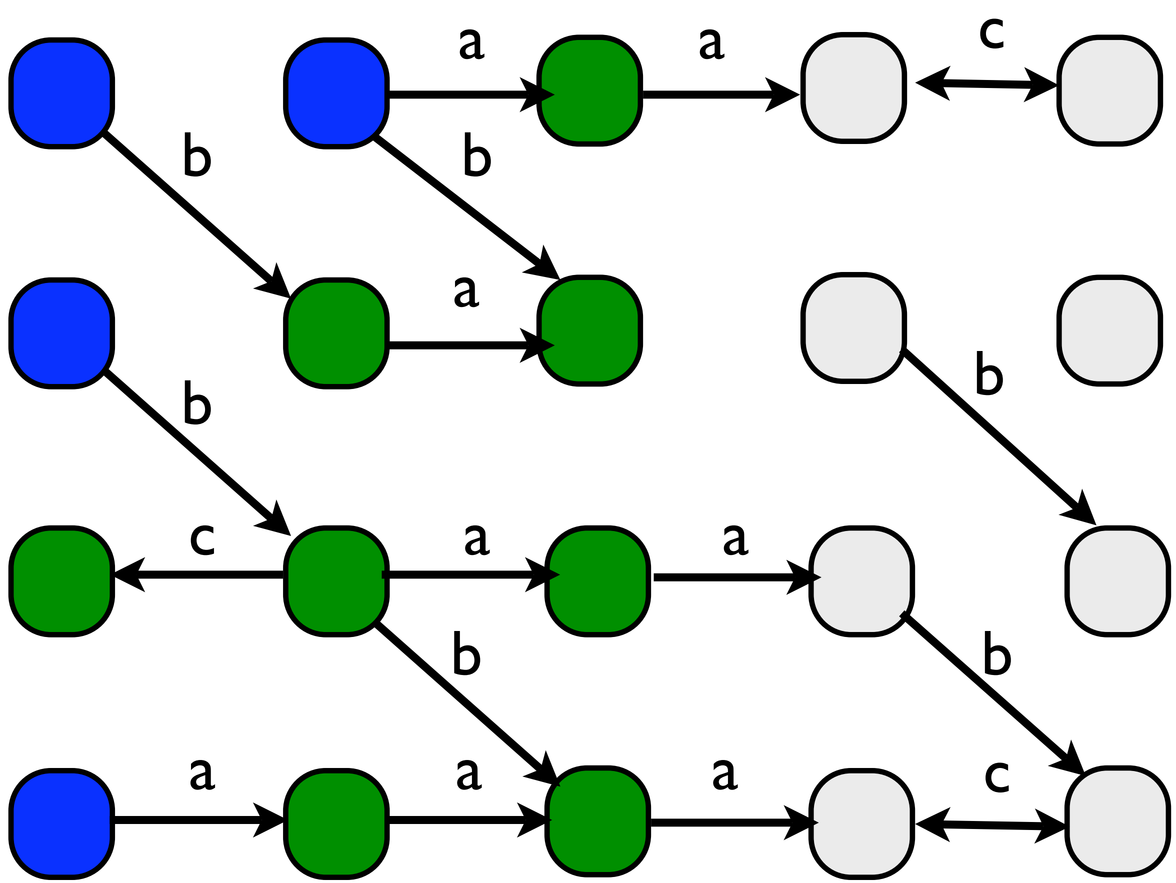
POST(X)=



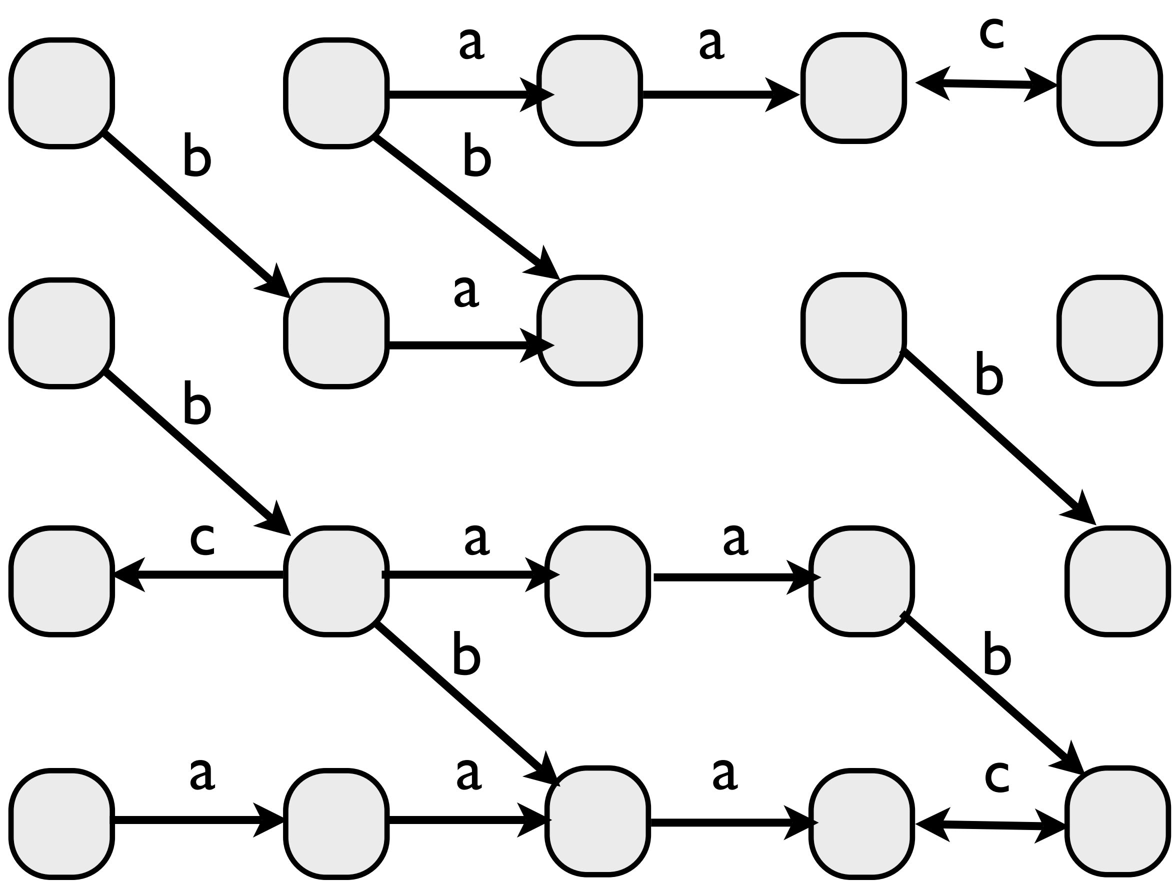
POST(X)=



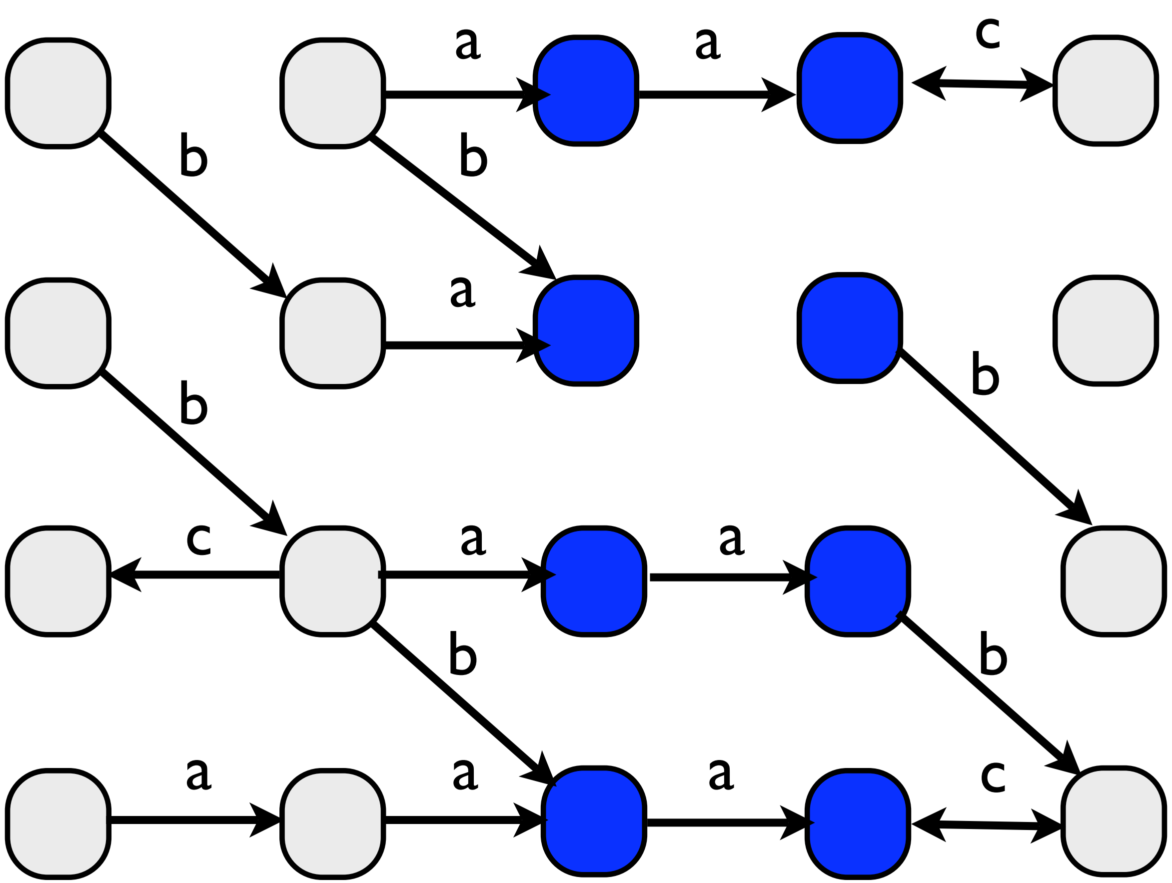
POST(X)=Y



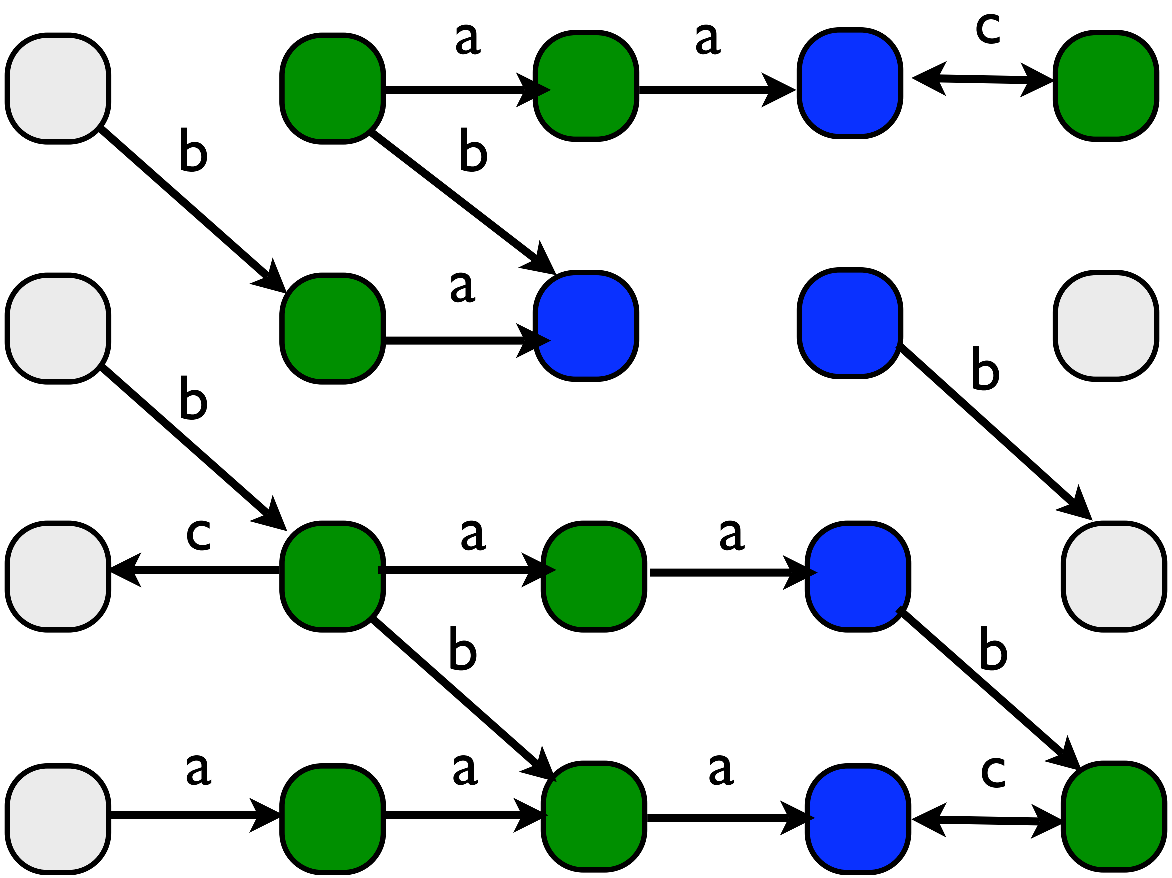
PRE(X)=



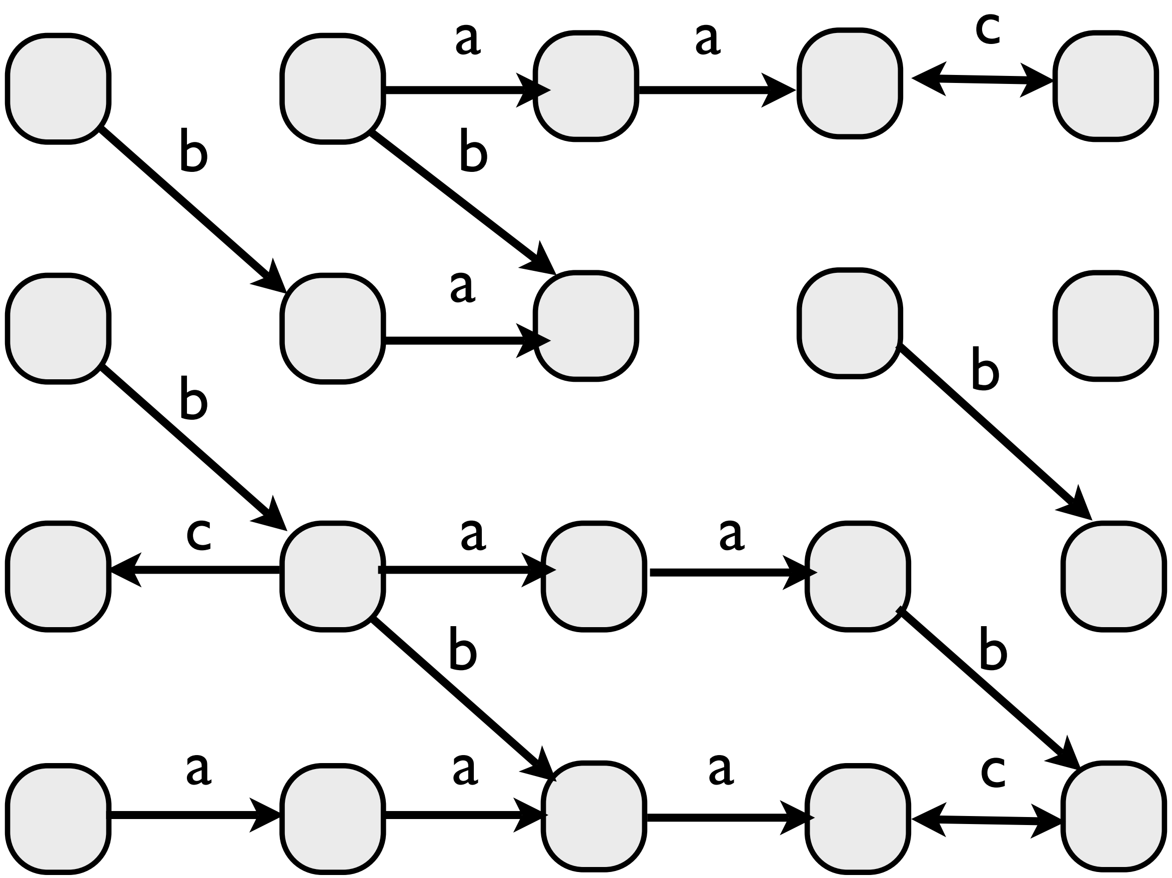
PRE(X)=



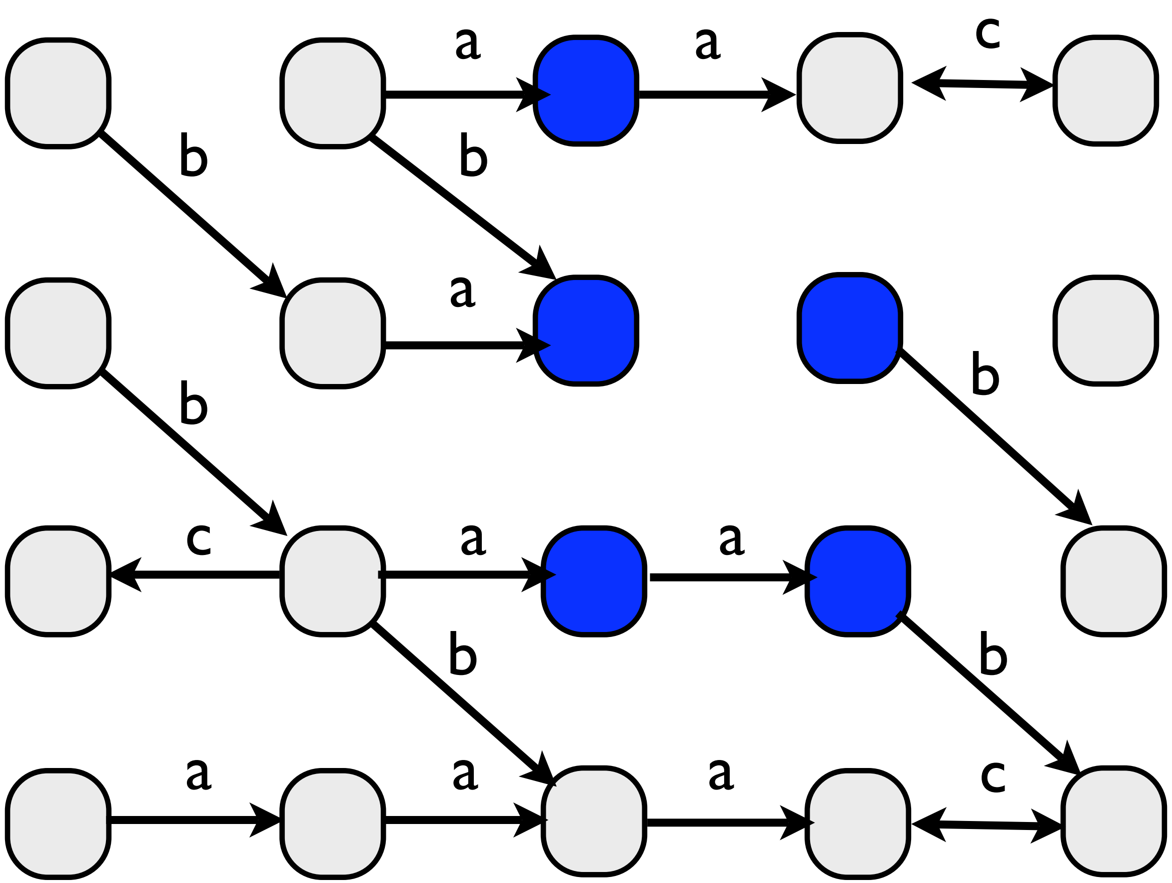
PRE(X)=Y



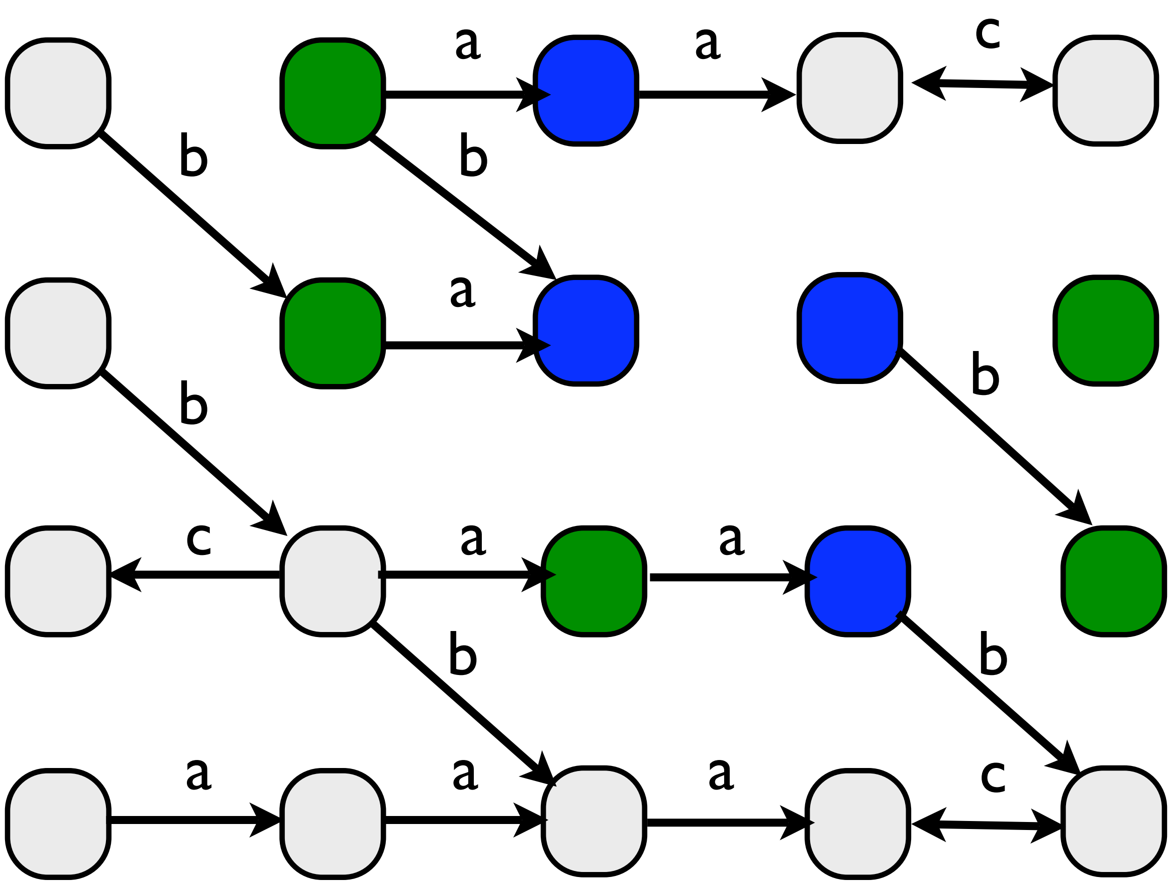
APRE(X)=



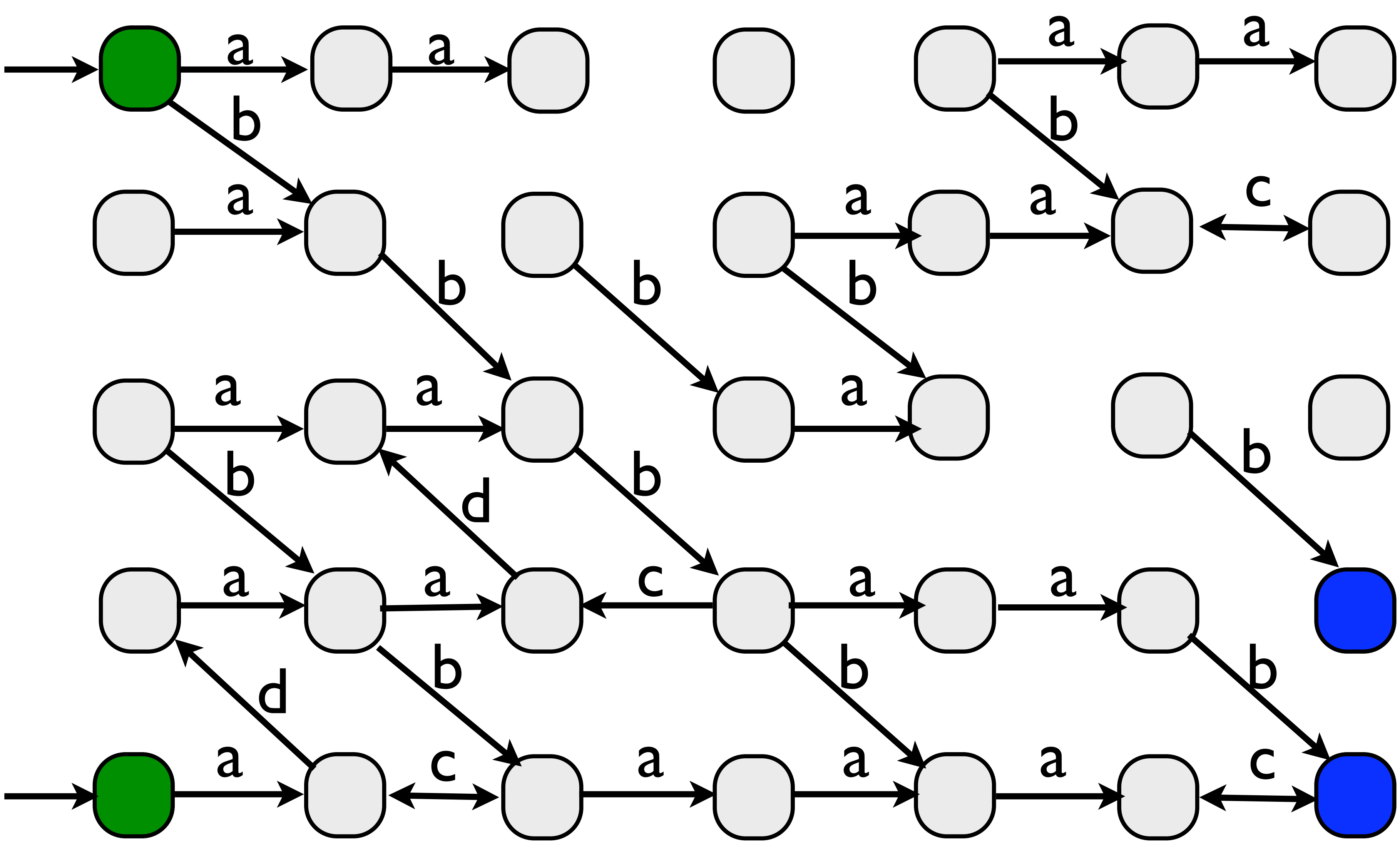
APRE(X)=



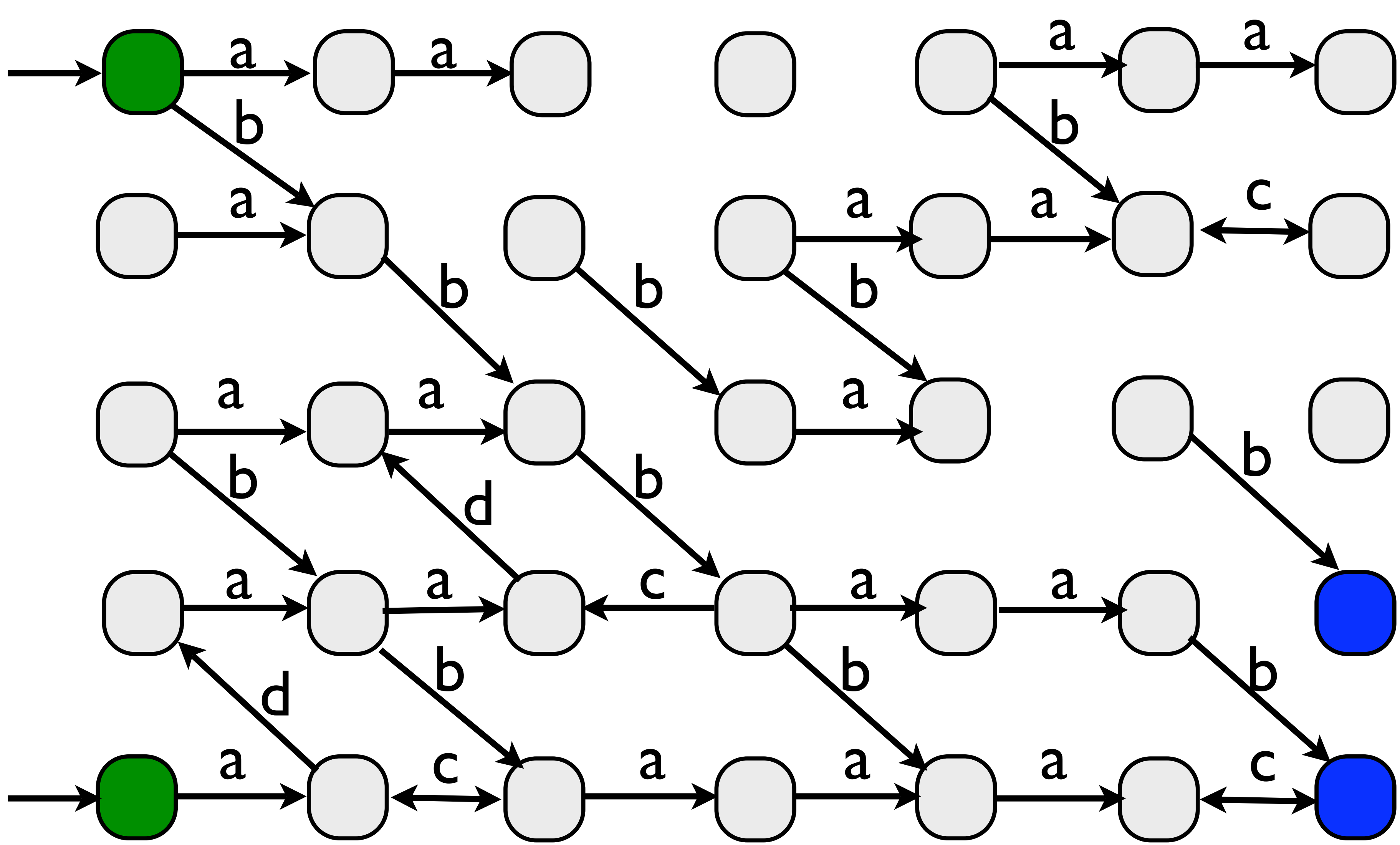
APRE(X)=Y



How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?



How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !



How can we use the **POST** operator to solve the following **reachability** question:
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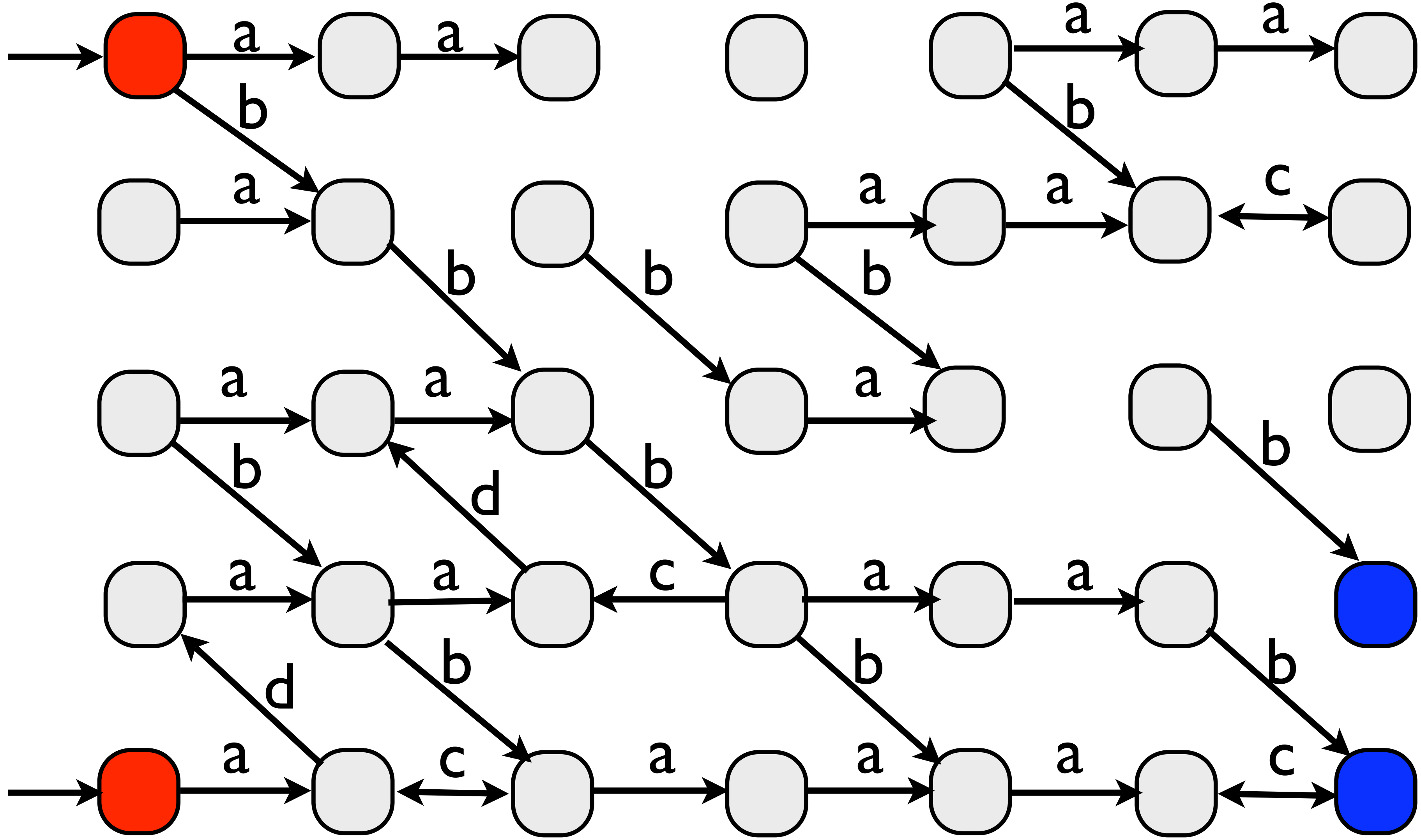
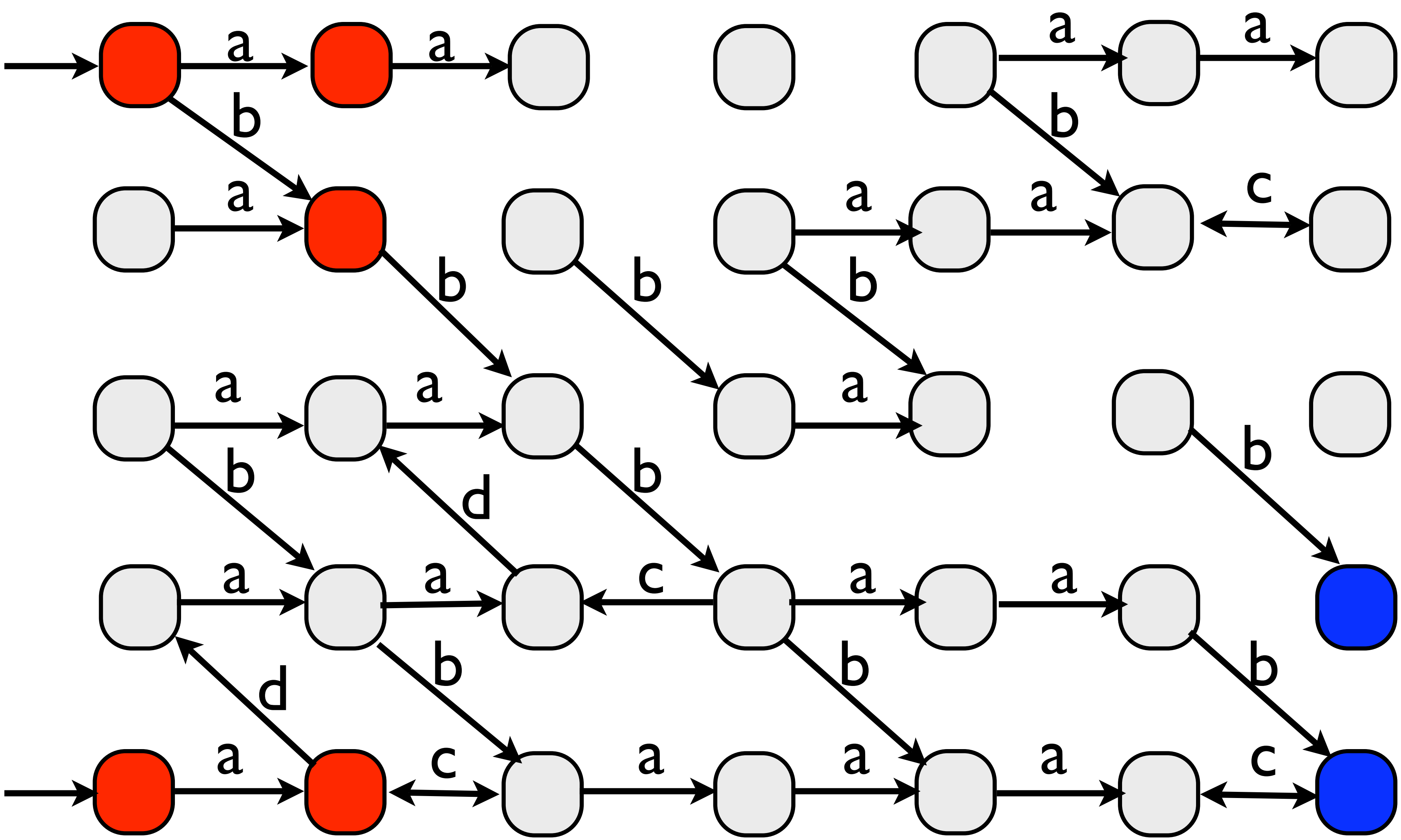
0 step



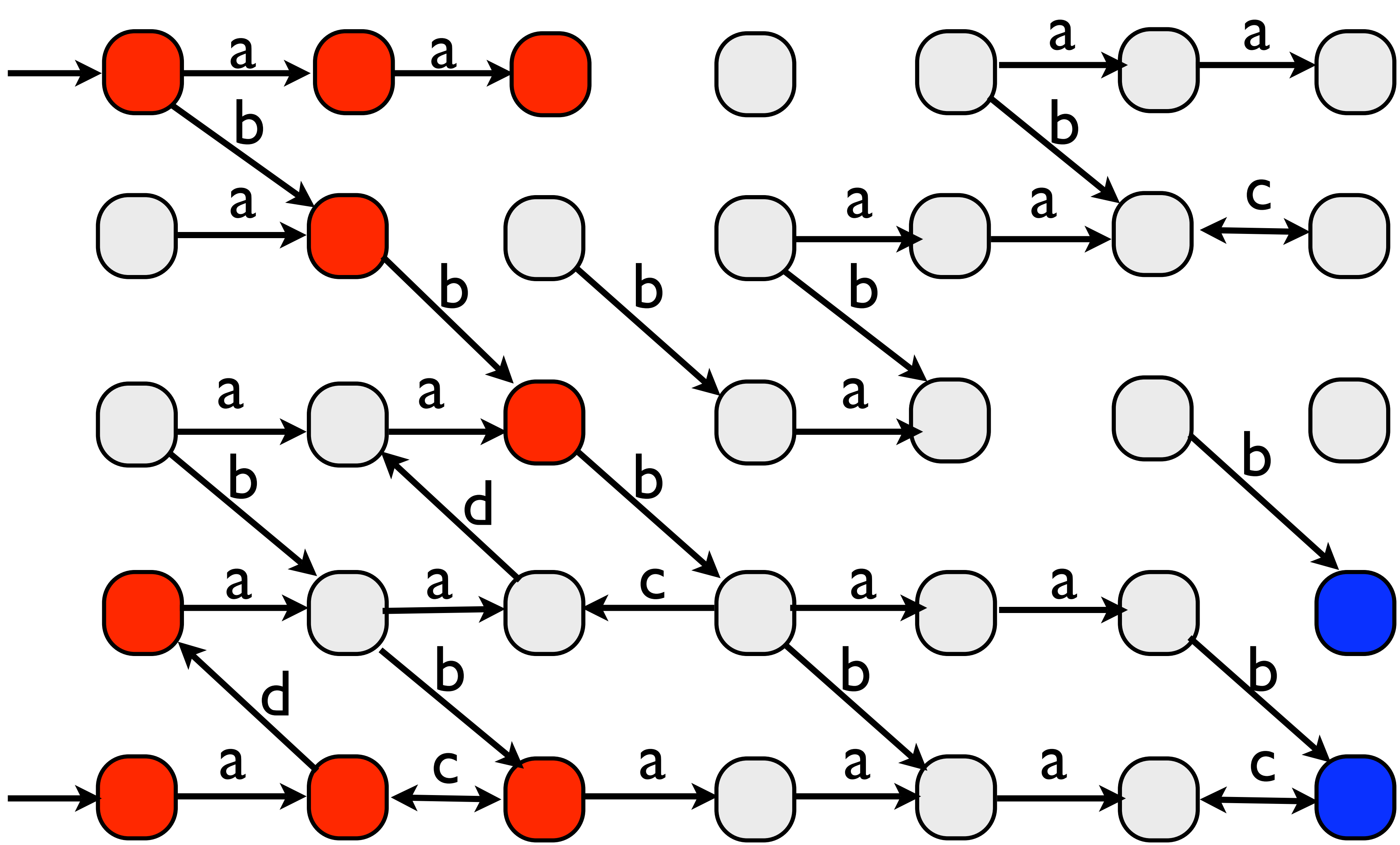
How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !

0 or 1 step



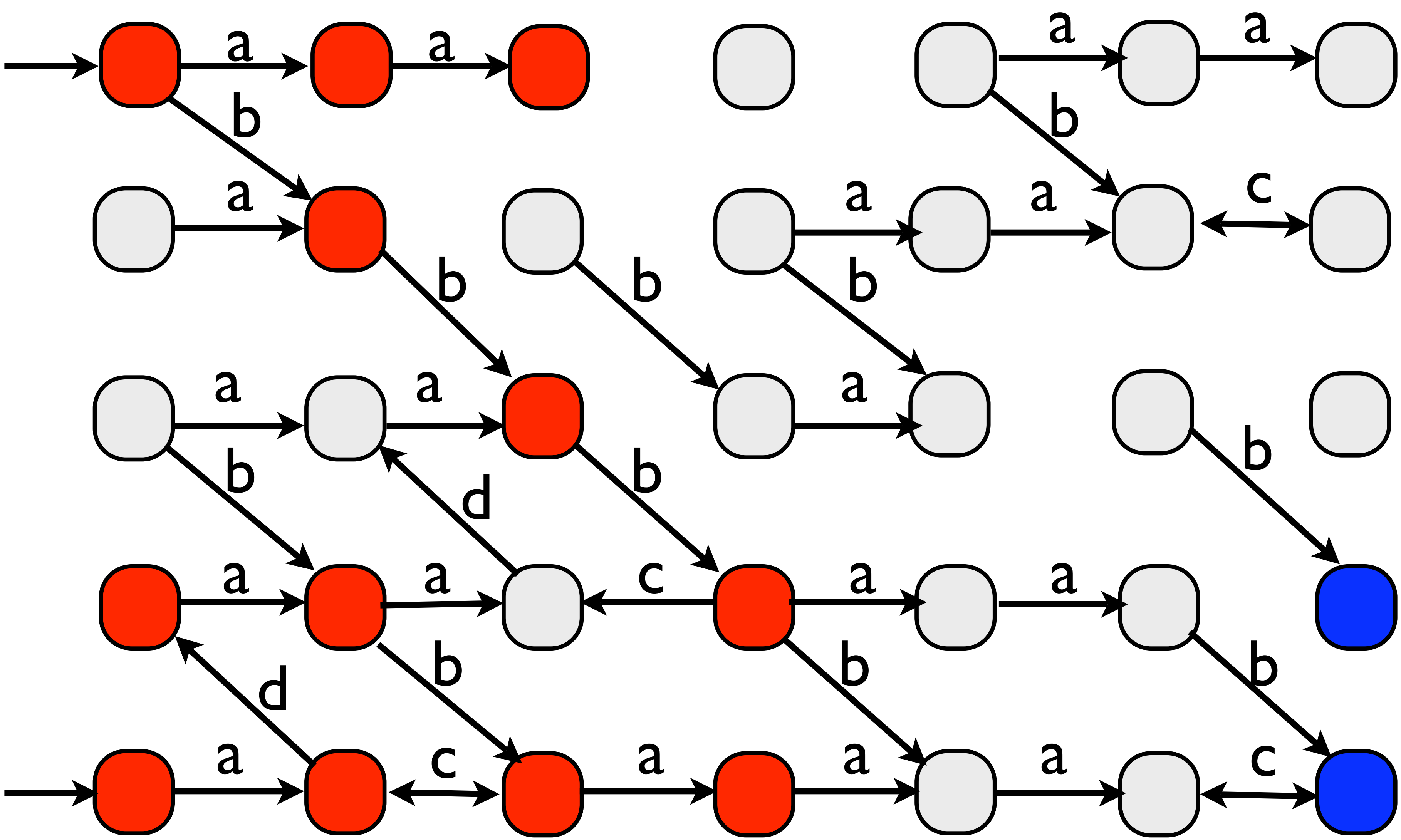
How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !

0, 1, 2 steps



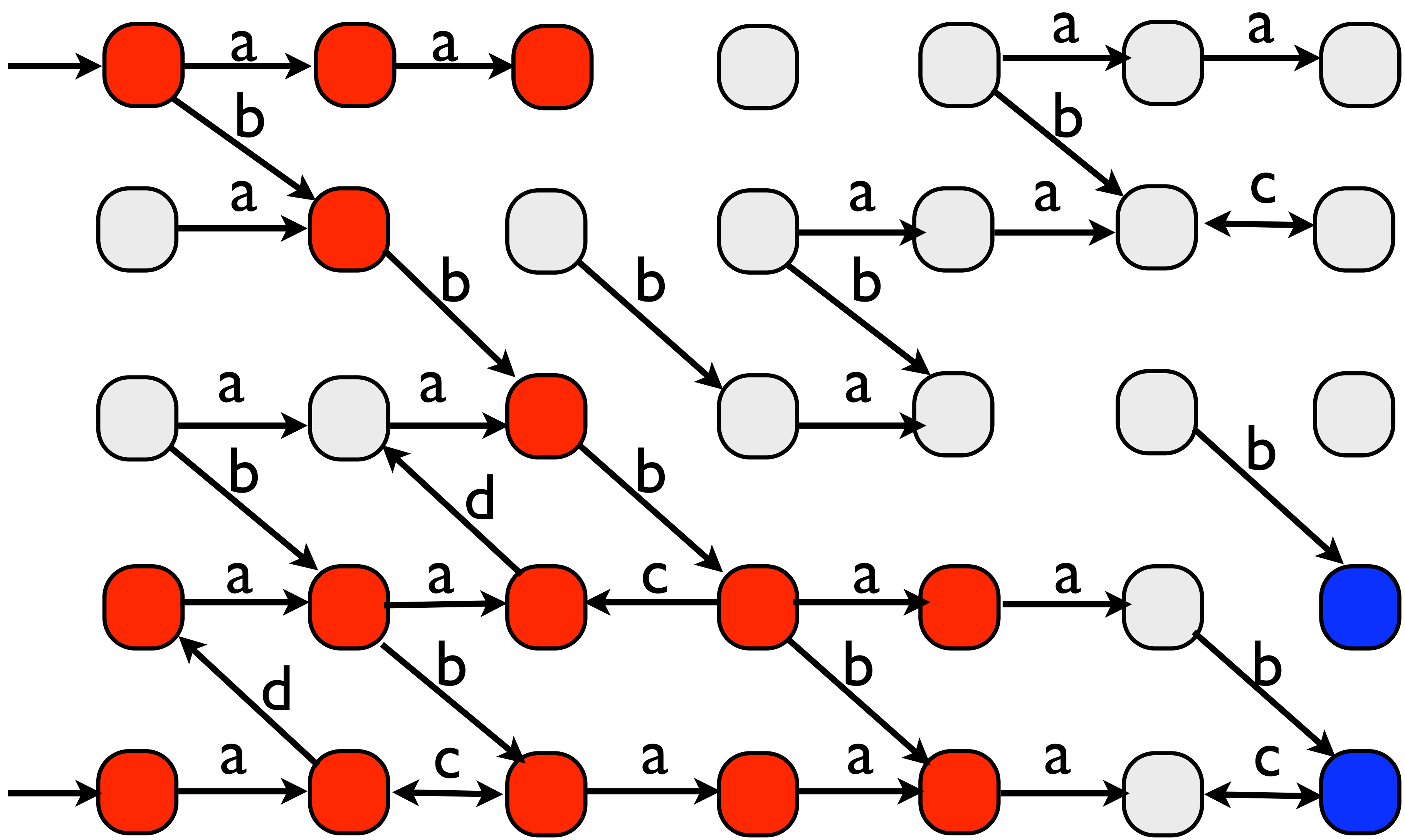
How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !

0, 1, 2, 3 steps



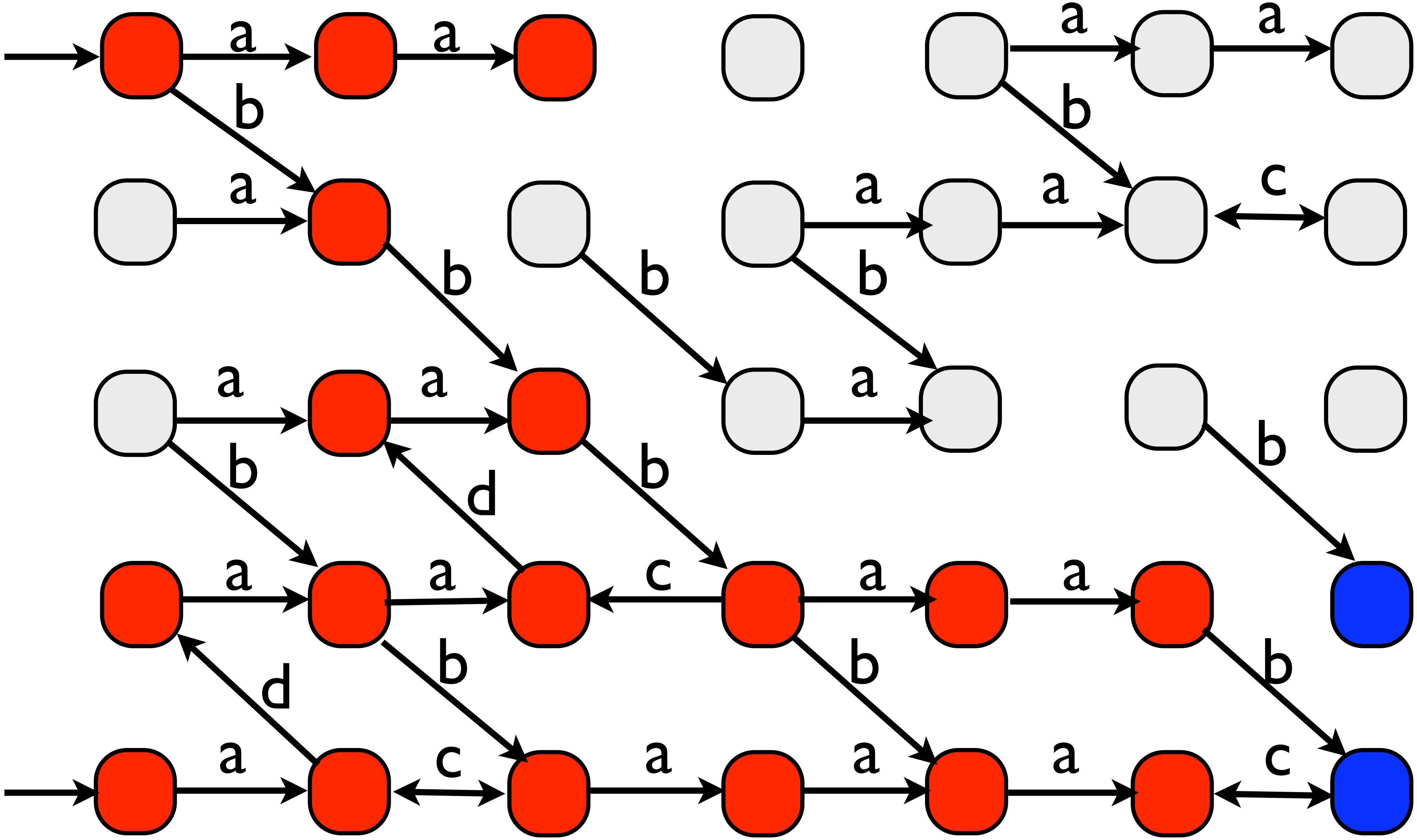
How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !

0,1, 2, 3, 4 steps



How can we use the **POST** operator to solve the following **reachability** question:
Can we reach the blue states from initial states ?
... By iterating the POST operator !

0,1, 2, 3, 4, 5 steps



How can we use the **POST** operator to solve the following **reachability** question:
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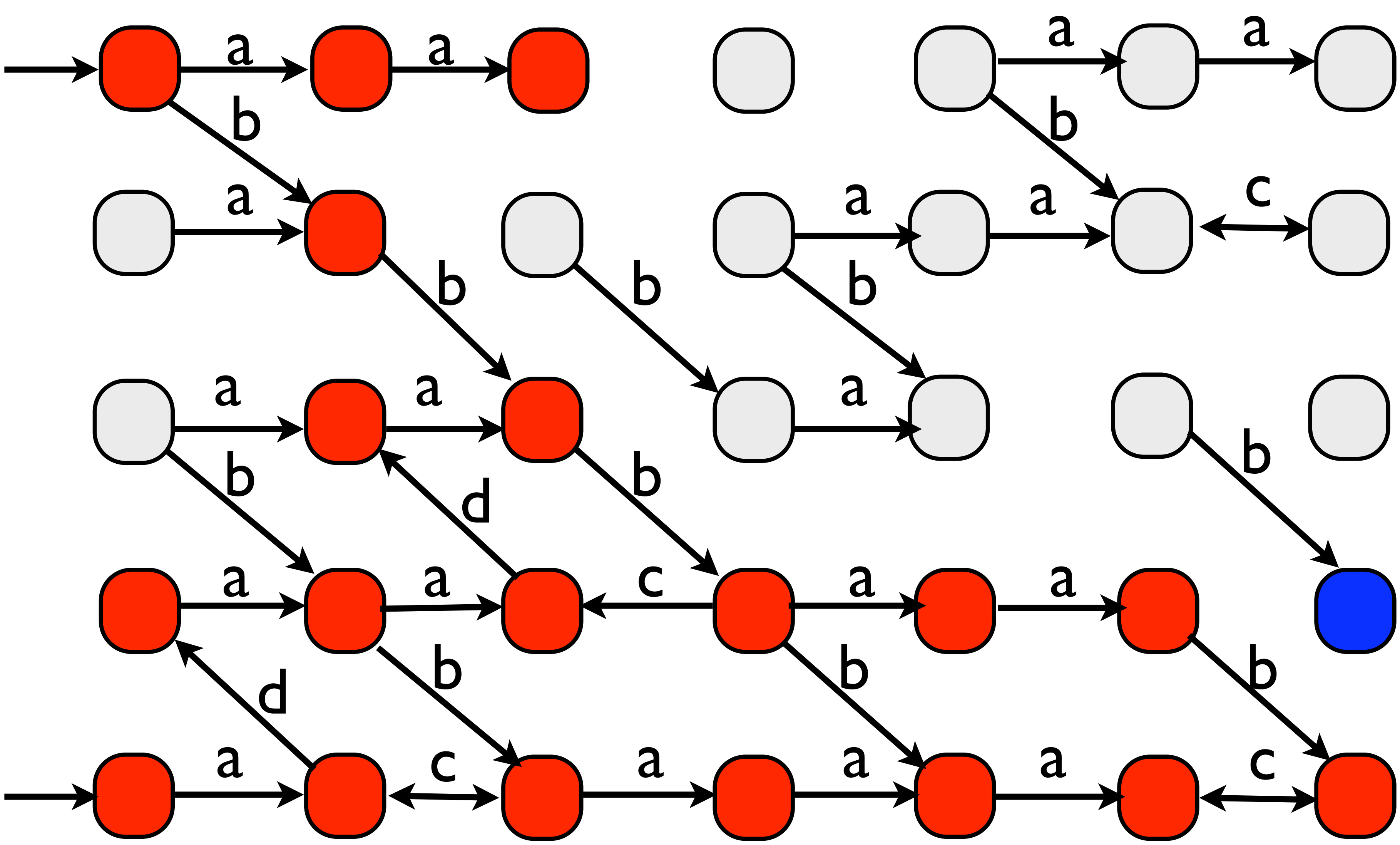
... By iterating the POST operator !

0,1, 2, 3, 4, 5, 6 steps



How can we use the **POST** operator to solve the following **reachability** question:
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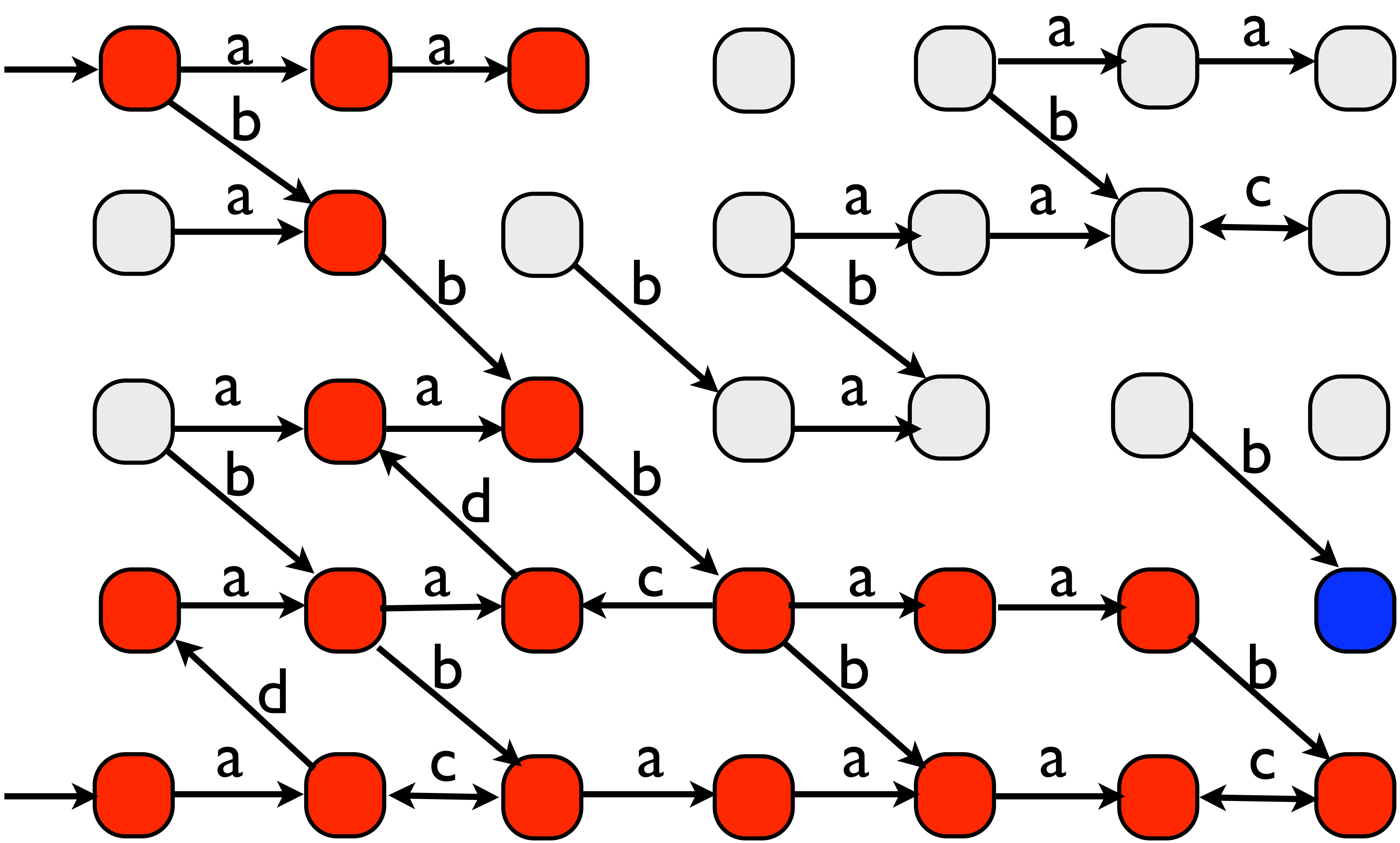
... By iterating the POST operator !

0, 1, 2, 3, 4, 5, 6, ∞ steps - we have reached a **fixed point**



Plan of the talk

- Labelled transition systems
- Properties of labeled transition systems:
  Reachability - Safety - Büchi properties
- Pre-Post operators
- Partial orders - Fixed points
- Symbolic model-checking
- Application to TA: region equivalence, region automata, zones

Partial orders

- Let **S** be a set. A **partial order** over **S** is a relation $\leq \subseteq \mathbf{S}\times\mathbf{S}$ such that the following properties hold:

  (*i*) reflexivity: $\forall s \in \mathbf{S}: s \leq s$,

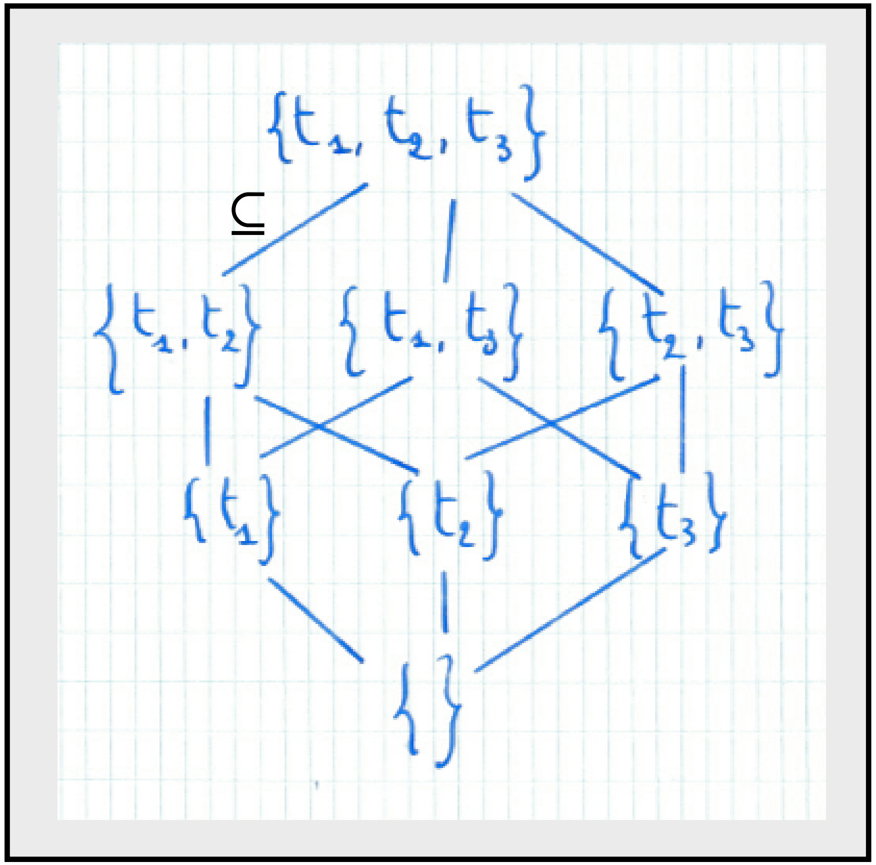
  (*ii*) transitivity: $\forall s_1, s_2, s_3 \in \mathbf{S}: s_1 \leq s_2 \wedge s_2 \leq s_3 \rightarrow s_1 \leq s_3$,

  (*iii*) antisymmetry: $\forall s_1, s_2 \in \mathbf{S}: s_1 \leq s_2 \wedge s_2 \leq s_1 \rightarrow s_1 = s_2$.

- A pair $(\mathbf{S}, \leq)$ such that $\leq$ is a partial order over **S** is called a **partially ordered set**.

- Let **T** be a set, we note $P(\mathbf{T})$ for the set of subsets of **T**. Example: if $\mathbf{T}=\{t_1,t_2,t_3\}$ then $P(\mathbf{T})=\big\{\{\},\{t_1\},\{t_2\},\{t_3\},\{t_1,t_2\},\{t_2,t_3\},\{t_1,t_3\},\{t_1,t_2,t_3\}\big\}$. Clearly, for any set **T**, $(P(\mathbf{T}),\subseteq)$ is a **partially ordered set**.

Graphical representation of P(**T**)



(P(**T**), ⊆) is a partially ordered set:

⊆ is reflexive: $\{\} \subseteq \{\}$, $\{t_1\}\subseteq\{t_1\}$,..., $\{t_1,t_2\}\subseteq\{t_1,t_2\}$, ...

⊆ is transitive: $\{t_1\}\subseteq\{t_1,t_2\}\wedge\{t_1,t_2\}\subseteq\{t_1,t_2,t_3\}\rightarrow\{t_1\}\subseteq\{t_1,t_2,t_3\}$

and clearly, ⊆ is antisymmetric.

Lower and upper bounds

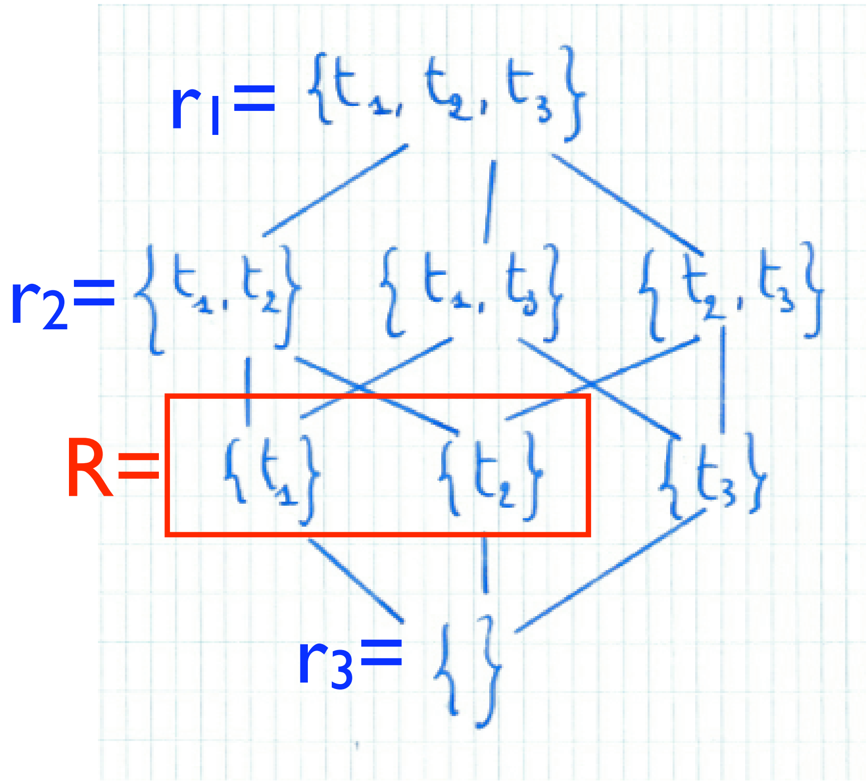
- Let (**S**,≤) be a partially ordered set. Let s∈**S** and **S**'⊆**S**,

  s is a **lower-bound** of **S**' iff ∀s'∈**S**'·s≤s'.

  s is a **upper-bound** of **S**' iff ∀s'∈**S**'·s'≤s.

- Let s be lower-bound for **S**', we say that s is the **greatest lower-bound** **(glb)** for **S**' iff for all lower-bound s' for **S**', we have s'≤s. We note **glb**(**S**') the **glb** of **S**' it it exists.

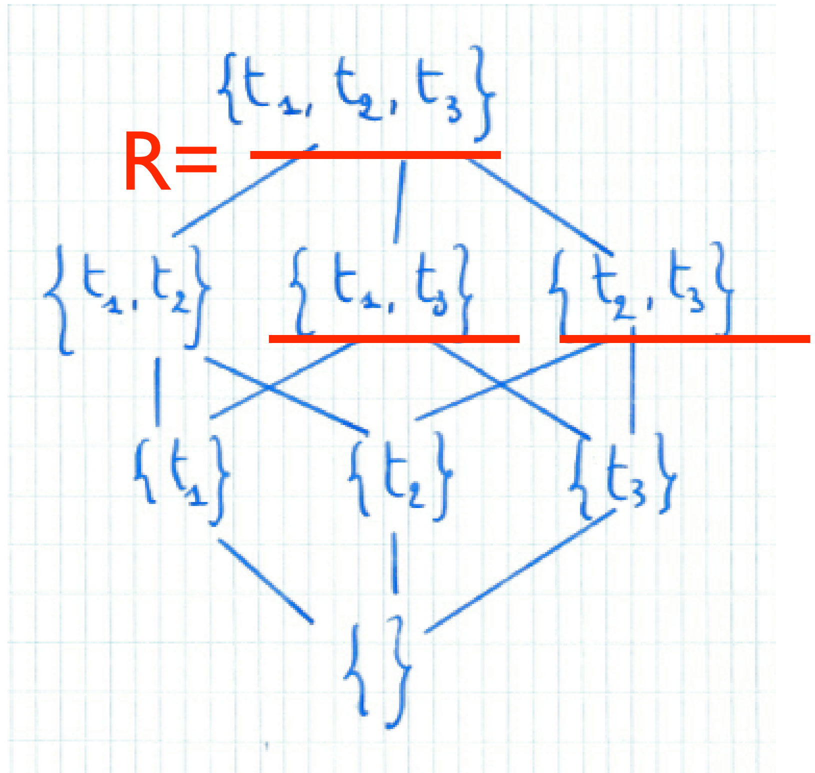
- Let s be upper-bound for **S**', we say that s is the **least upper-bound** **(lub)** for **S**' iff for all upper-bound s' for **S**', we have s≤s'. We note **lub**(**S**') the **lub** of **S**' it it exists.



➪ r_1 and r_2 are **upper bounds** of R.

➪ r_2 is the **least upper bound** of R.

➪ r_3 is the only **lower-bound** of R and so it is the **greatest lower bound** of R.



The **lub** of a set of sets R_i is equal to $\cup_i R_i$

ex: **lub** R = $\{t_1,t_2,t_3\}$

The **glb** of a set of sets R_i is equal to $\cap_i R_i$

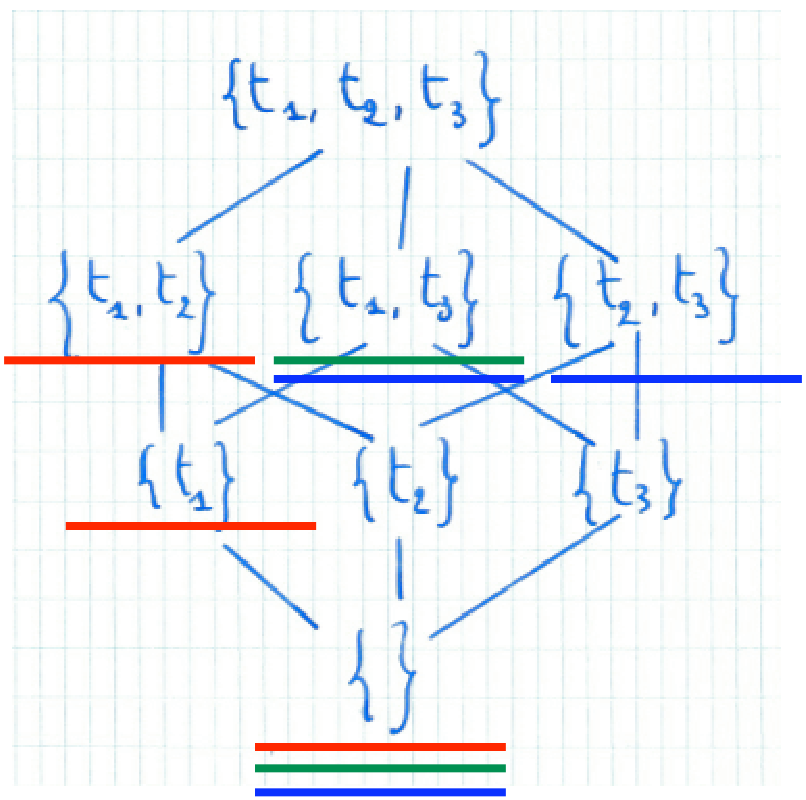
ex: **glb** R = $\{t_3\}$

A set of elements **S**'⊆**S** is a **chain**

iff

$\forall s,s' \in \mathbf{S}' \cdot s \leq s' \vee s' \leq s$

i.e. all pairs of elements are ordered by ≤
(increasing sequence of elements).



R={{},{t_1},{t_1,t_2}} is a chain in P(**T**).

R'={{},{t_1,t_3}} is a chain in P(**T**).

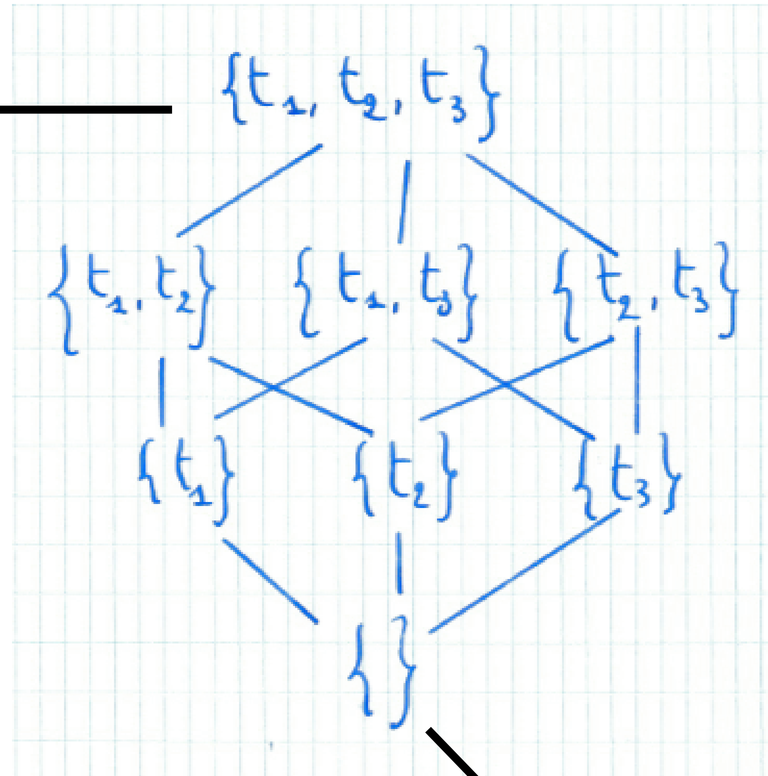
R''= {{},{t_1,t_3},{t_2,t_3}} is not a chain in P(**T**).

- A partially ordered set (**S**,$\leq$) is a **complete partial order** if every chain in **S** has a **lub** in **S**.

- A complete partial order (**S**,$\leq$) is a **complete lattice** if every subset **S'** of **S** has a **lub** in (**S**,$\leq$).

- Note that **glb S'**= **lub**$\{s \in \mathbf{S} \mid \forall s' \in \mathbf{S}: s \leq s'\}$, so every subset **S'** in a complete lattice has also a **glb**.

- Example: (**P**(**T**),$\subseteq$) is a complete lattice. Indeed, remember that the **lub** of a set of sets R_i is equal to $\cup_i R_i$ and the **glb** of a set of sets R_i is equal to $\cap_i R_i$.



The maximal element of P(**T**),
i.e. the least upper-bound of P(**T**),
noted **Max**(P(**T**)).

The minimal element of P(**T**),
i.e. the greatest lower-bound of P(**T**),
noted **Min**(P(**T**)).

Min and **Max** elements in a complete lattice

- Let ($\mathbf{S}$,≤) be <u>a partially ordered set</u>. A function $\mathbf{f}$:$\mathbf{S}\rightarrow\mathbf{S}$ is **monotone** (order preserving) iff $\forall s,s'\in\mathbf{S}\cdot s\leq s'\rightarrow\mathbf{f}(s)\leq\mathbf{f}(s')$.

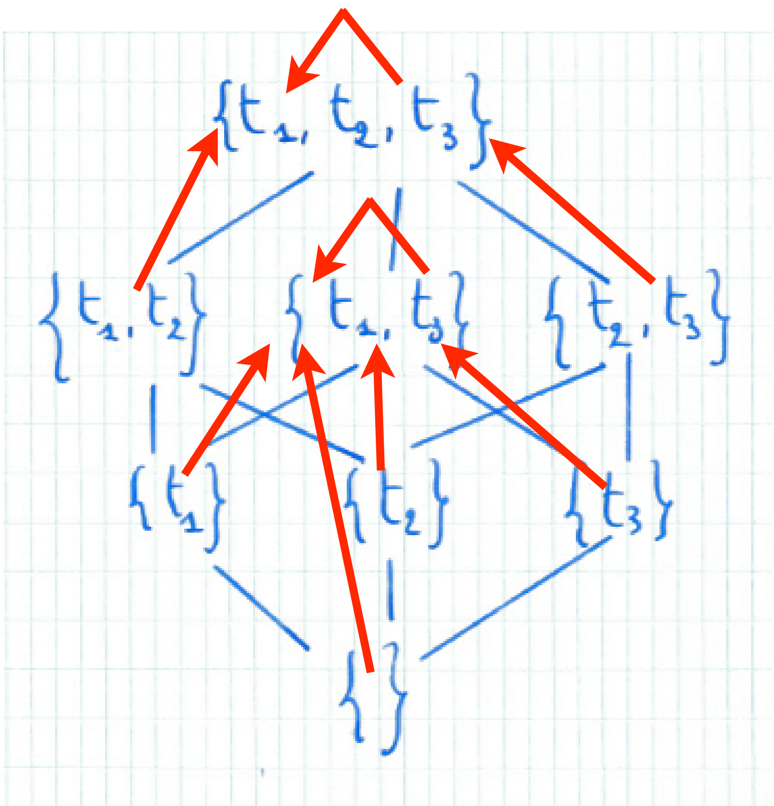
- Let ($\mathbf{S}$,≤) be a <u>complete partially ordered set</u>. A function $\mathbf{f}$:$\mathbf{S}\rightarrow\mathbf{S}$ is **continuous** iff $\mathbf{f}$ is monotone and for all non-empty chain $\mathbf{S}'$ in $\mathbf{S}$: $\mathbf{f}(\mathbf{lub}(\mathbf{S}'))=\mathbf{lub}(\mathbf{f}(\mathbf{S}'))$.

  Remark. In any finite complete partially ordered set $\mathbf{S}$, if $\mathbf{f}$ is monotone then $\mathbf{f}$ is continuous.

- $s\in\mathbf{S}$ is a **fixed point** of $\mathbf{f}$:$\mathbf{S}\rightarrow\mathbf{S}$ if $\mathbf{f}(s)=s$. The set of fixed points of $\mathbf{f}$ is noted $\mathbf{Fx}(\mathbf{f})$.

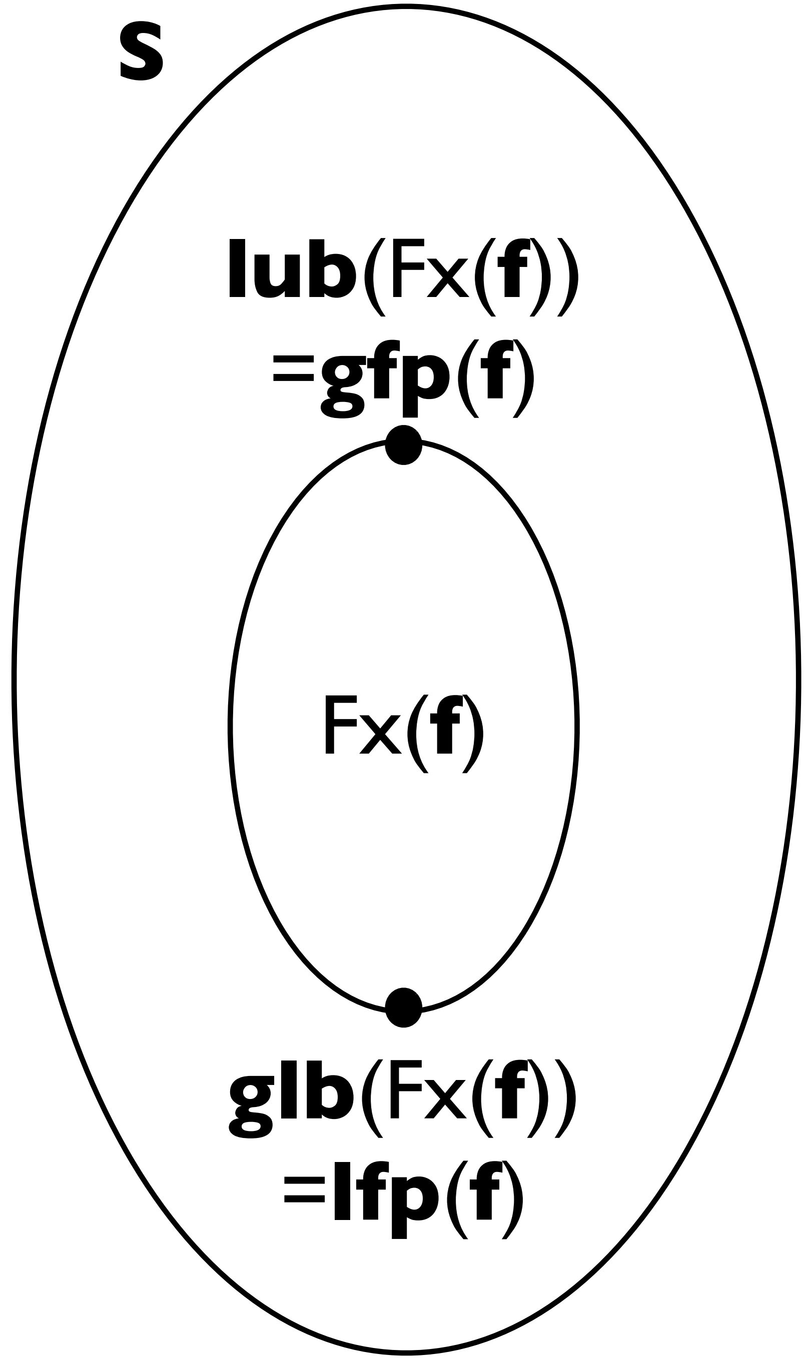
- **<u>Theorem</u>** (Tarski) Any monotone function on a **complete lattice** has a:
  (*i*) **least fixed point**, **lfp**(**f**), equal to **glb**(**Fx**(**f**))
  (*ii*) **greatest fixed point**, **gfp**(**f**), equal to **lub**(**Fx**(**f**))

Let us consider **f** as depicted by red arrows



Clearly, **f** is monotone (and so continuous).

Fx(**f**)={{t_1,t_3},{t_1,t_2,t_3}}.
lfp(**f**)={t_1,t_3}=**glb**(**Fx**(**f**)).
gfp(**f**)={t_1,t_2,t_3}=**lub**(**Fx**(**f**)).


S
lub(Fx(f))
=gfp(f)
Fx(f)
glb(Fx(f))
=lfp(f)

- Let $\mathbf{f}^i$ be defined **inductively** as
  - for i=0: $\mathbf{f}^0=\mathbf{f}$
  - for all i>0: $\mathbf{f}^i=\mathbf{f}(\mathbf{f}^{i-1})$.

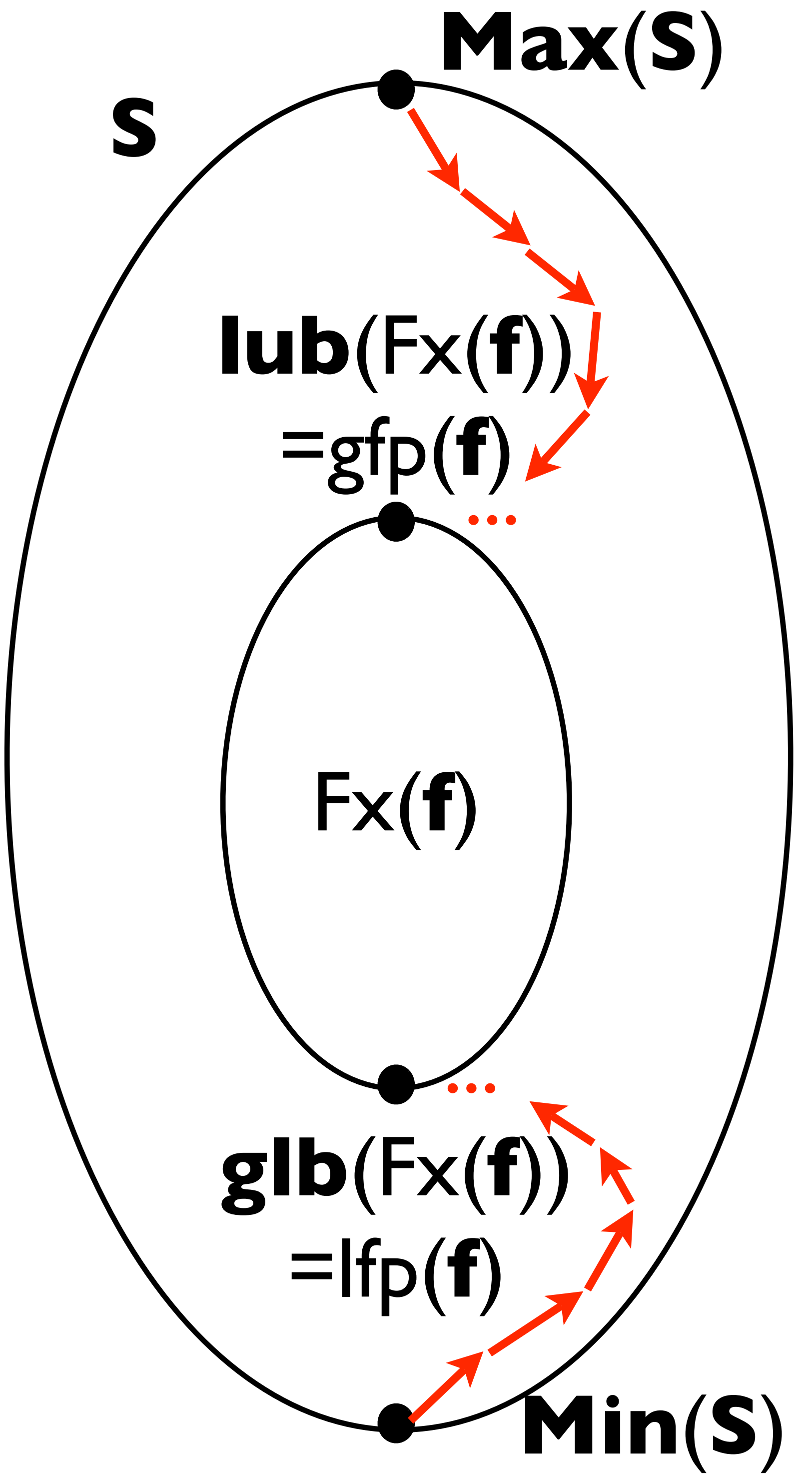
- **Theorem** (Kleene-Tarski) Let ($\mathbf{S}$,$\leq$) be a complete lattice, let $\mathbf{f}$:$\mathbf{S}\rightarrow\mathbf{S}$ be a continuous:

  $\mathbf{lfp}(\mathbf{f})=\mathbf{glb}\ \{\ \mathbf{f}^i(\mathbf{Min}(S))\ |\ i\geq 0\ \}$ and

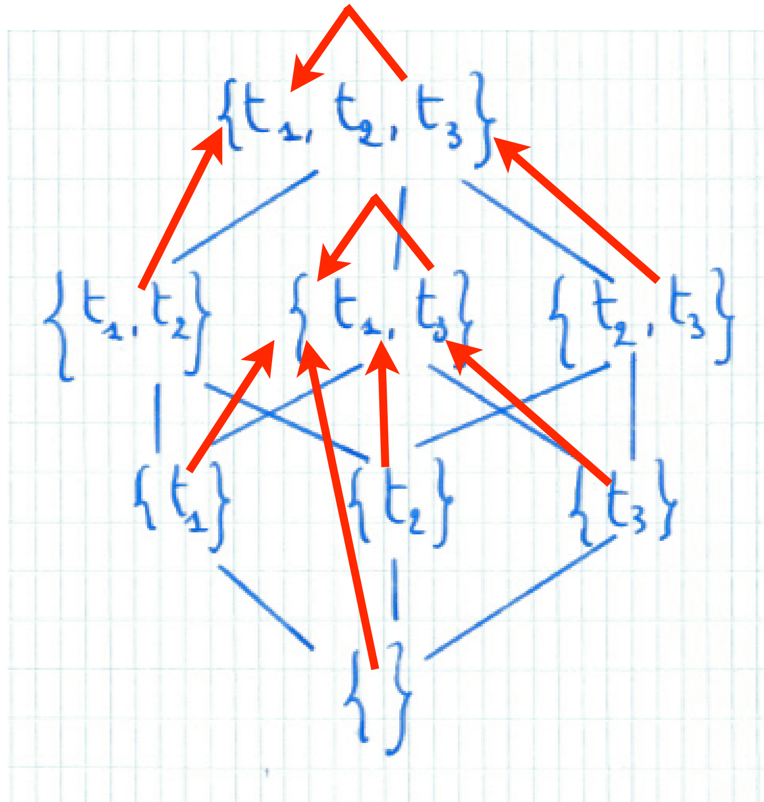
  $\mathbf{gfp}(\mathbf{f})=\mathbf{lub}\ \{\ \mathbf{f}^i(\mathbf{Max}(S))\ |\ i\geq 0\ \}$.

- This gives us an **iterative schema** to compute the **lfp**(**f**) (**gfp**(**f**)) of a continuous function **f**:

  ⇨ iterate the function from the **Min** (**Max**) of the set until stabilization.



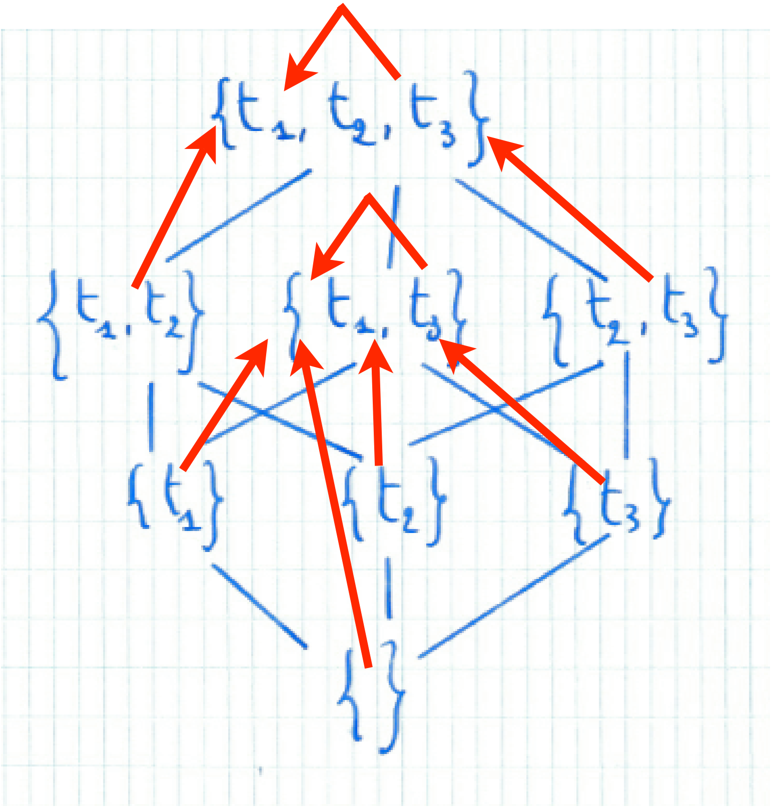
Max(S)
S
lub(Fx(f))
=gfp(f)
Fx(f)
glb(Fx(f))
=lfp(f)
Min(S)



Computation of **lfp**(**f**)

$\mathbf{f}^0(\{\})=\{t_1,t_2\}$

$\mathbf{f}^1(\{\})=\mathbf{f}(\{t_1,t_2\})=\{t_1,t_2\}=\mathbf{lfp}(\mathbf{f})$



Computation of **gfp**(**f**)

$\mathbf{R}_0 = \mathbf{f}^0(\{t_1,t_2,t_3\}) = \{t_1,t_2,t_3\} = \mathbf{gfp}(\mathbf{f})$

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Symbolic model-checking

- The reachability, safety, and Büchi objectives can be solved using **fixed point equations**.
- Solving those equations will be done by iteration of functions built from the **Pre**, **Apre** or **Post** operators on sets of states.
- Those algorithms are called **symbolic** because they **manipulate sets** of states directly instead of manipulating **individual states** as it is done in so-called **explicit** model-checking algorithms.

Fixed points for reachability

- Let us consider an instance of the **reachability problem** given by the LTS $L=(S,S_0,\Sigma,T,C,\lambda)$, and a set of states **Goal** $\subseteq S$;
- **Goal** is reachable in the LTS

  iff $\mathbf{lfp}\ (\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)) \cap \mathbf{Goal} \neq \emptyset$

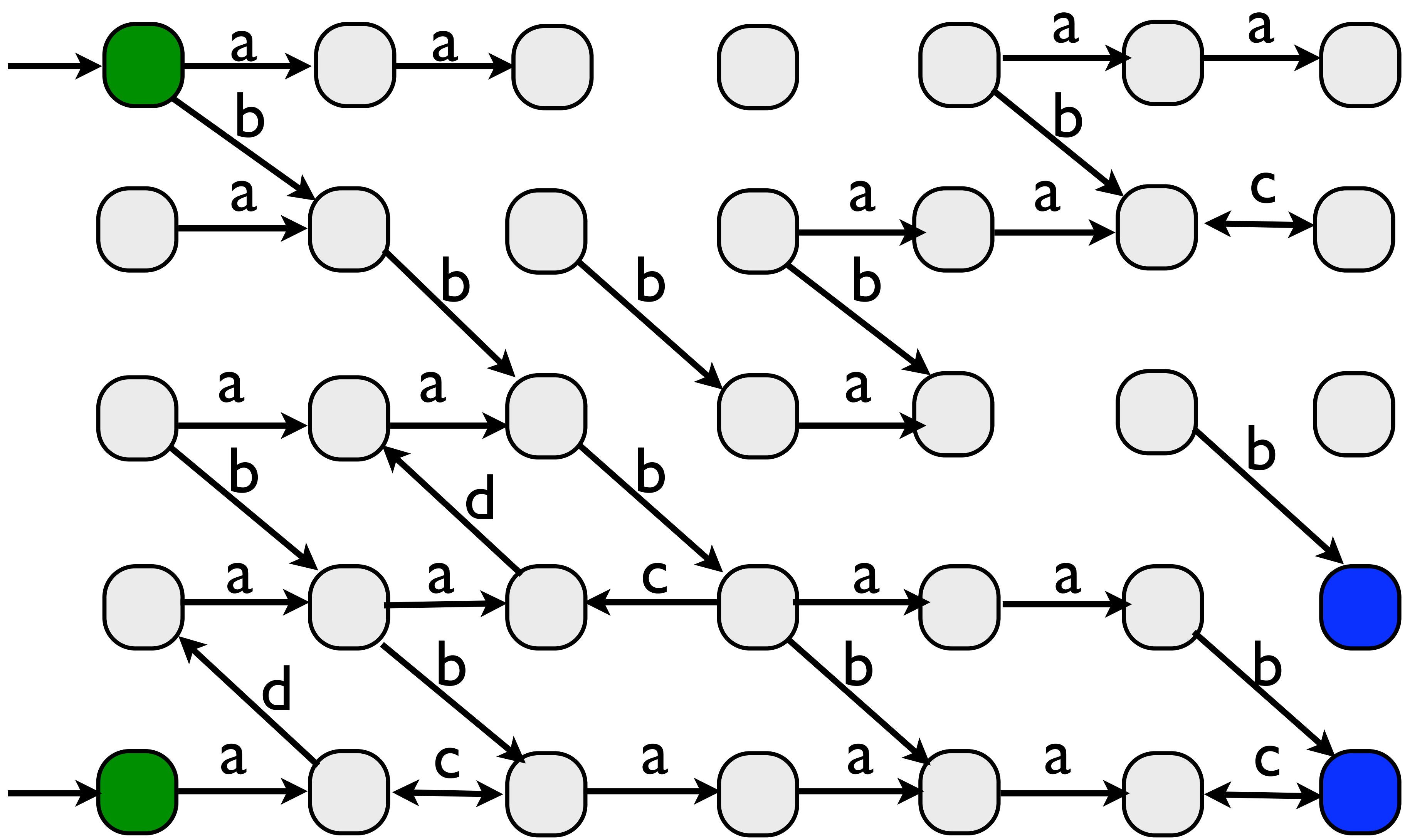
  this is a forward algorithm

  iff $\mathbf{lfp}\ (\lambda X.\ \mathbf{Goal} \cup \mathbf{PRE}(X)) \cap \mathbf{S_0} \neq \emptyset$

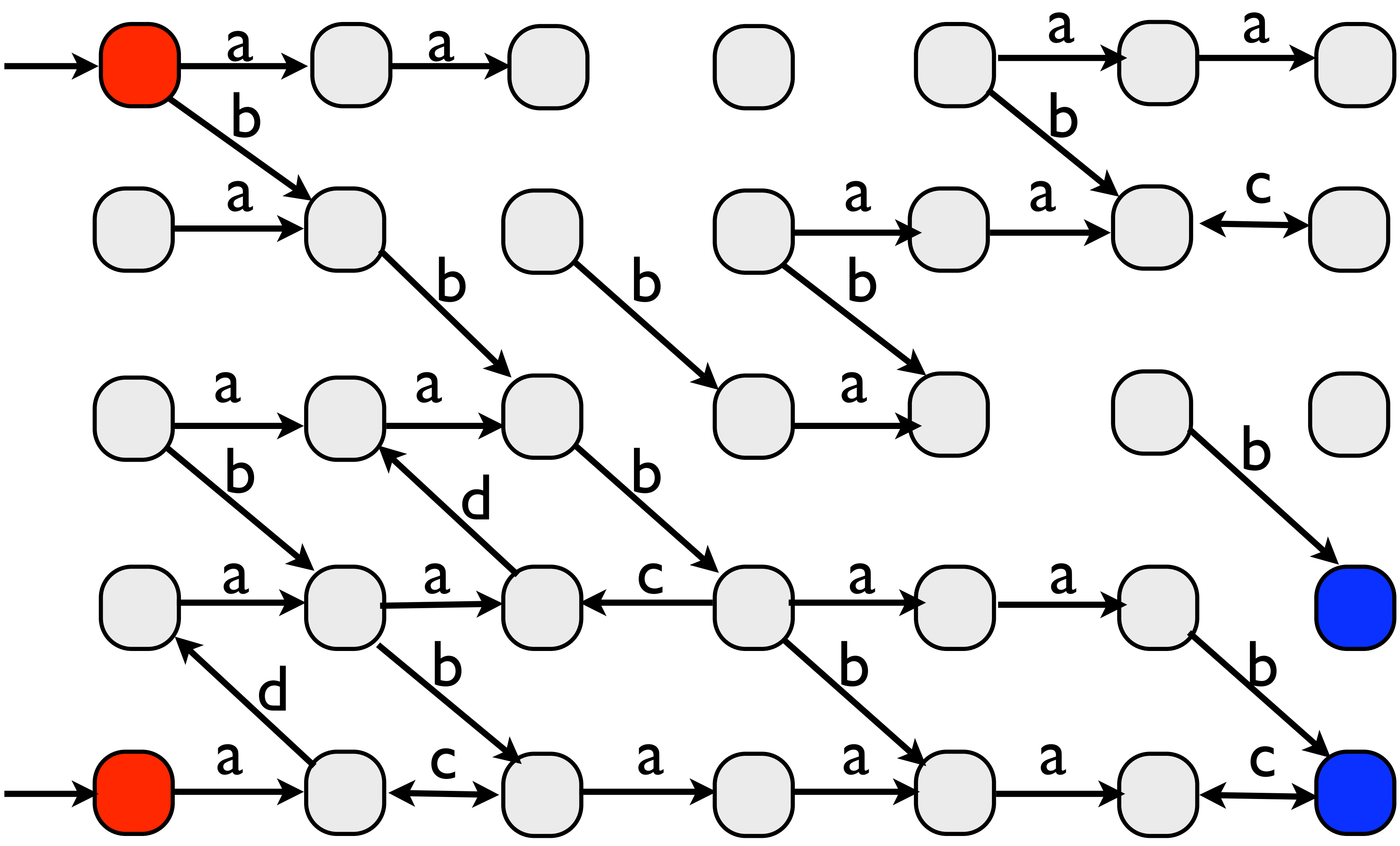
  this is a backward algorithm

Reachability - Forward algorithm

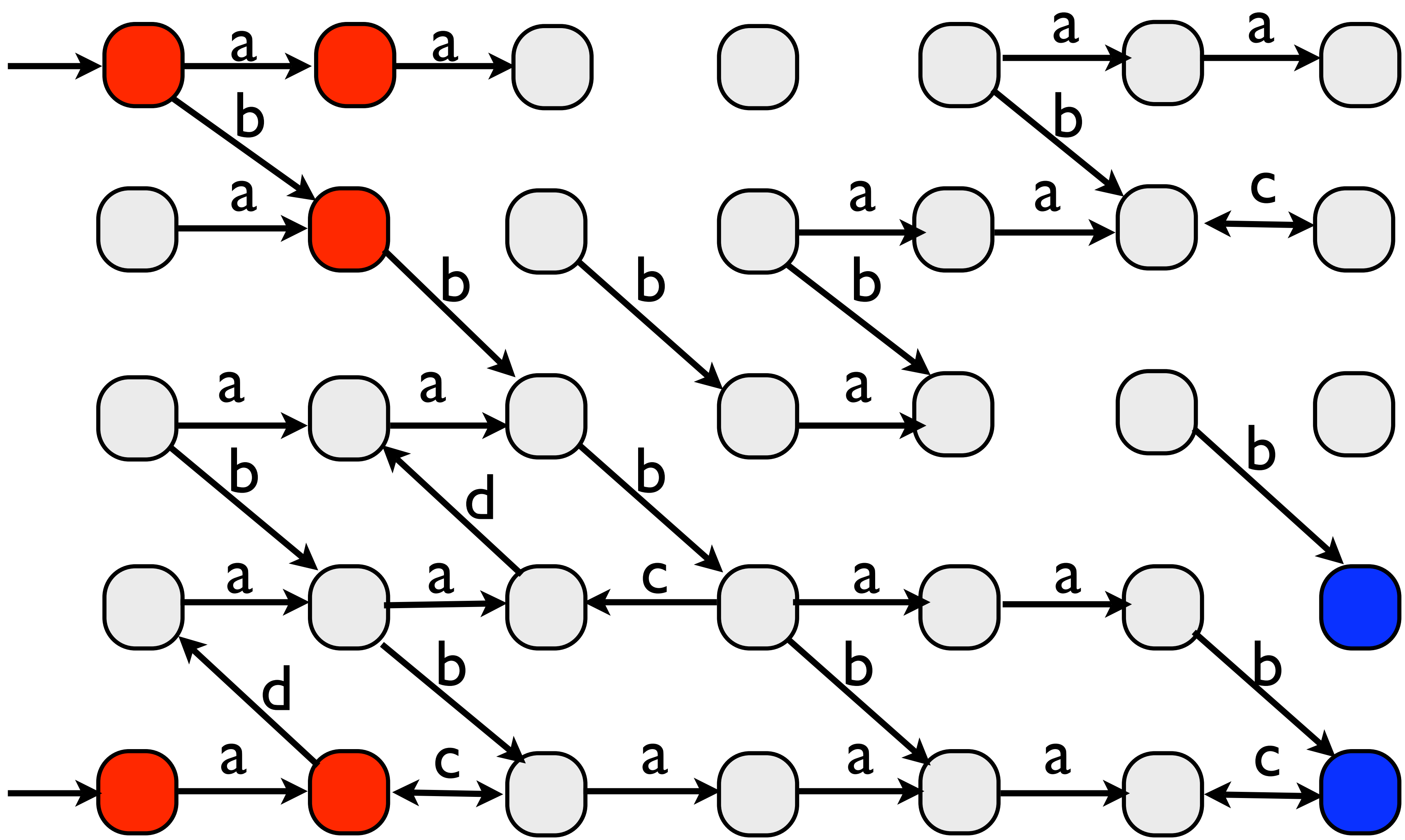
$\mathbf{lfp}\ (\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)) \cap \mathbf{Goal} \neq \emptyset$



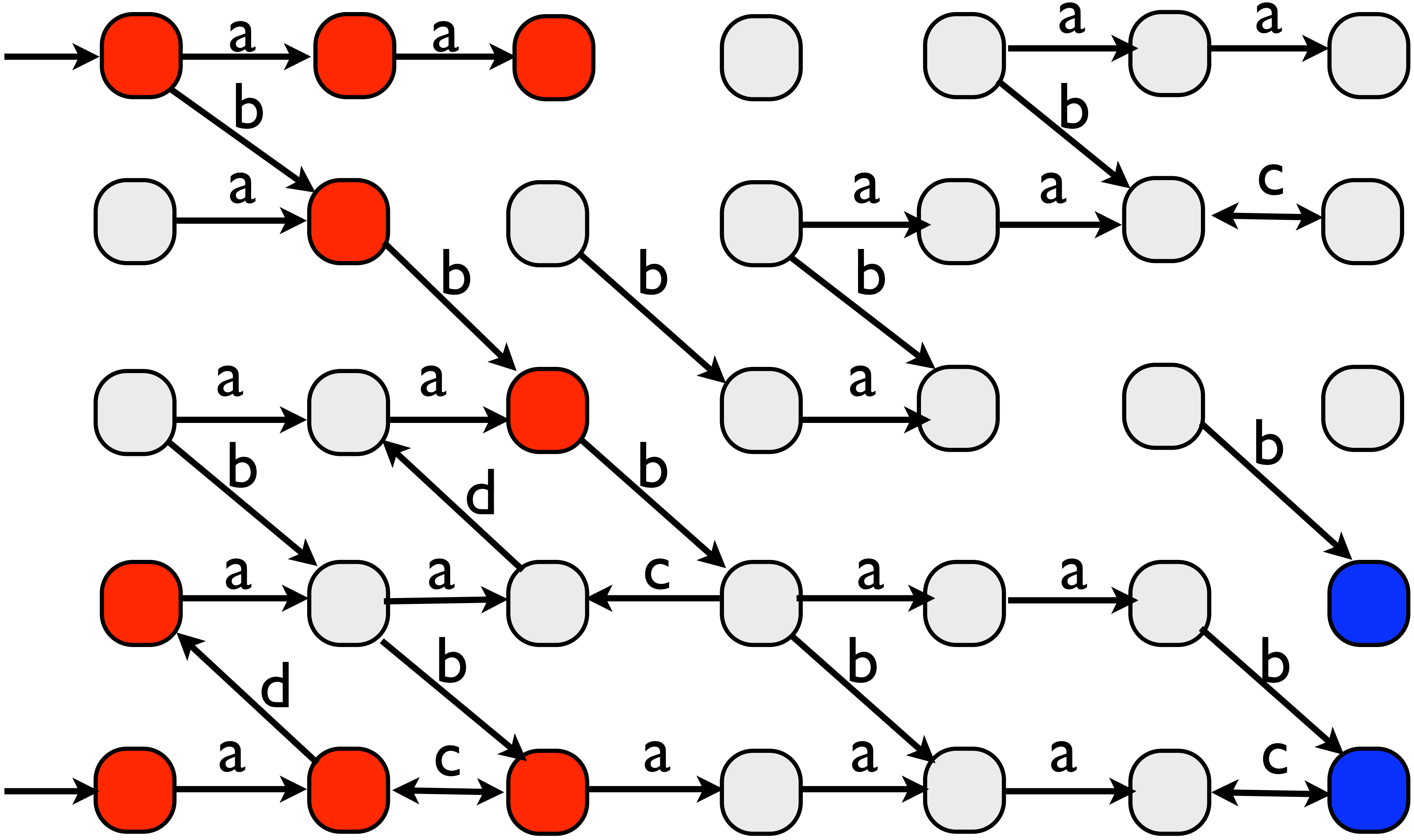
Iterative evaluation of **lfp** ($\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)$)



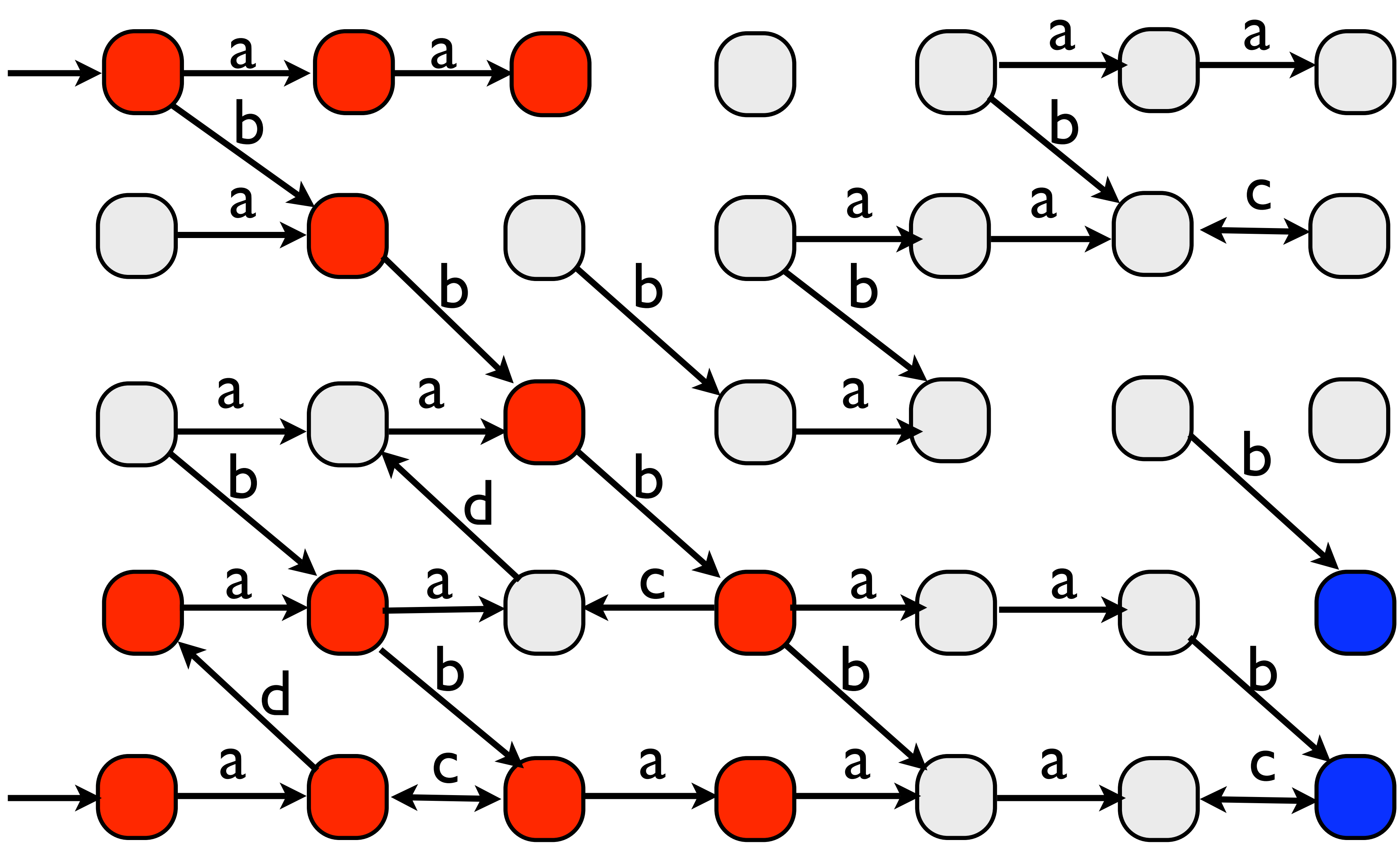
Iterative evaluation of **lfp** ($\lambda X.\, \mathbf{S_0} \cup \mathbf{POST}(X)$)



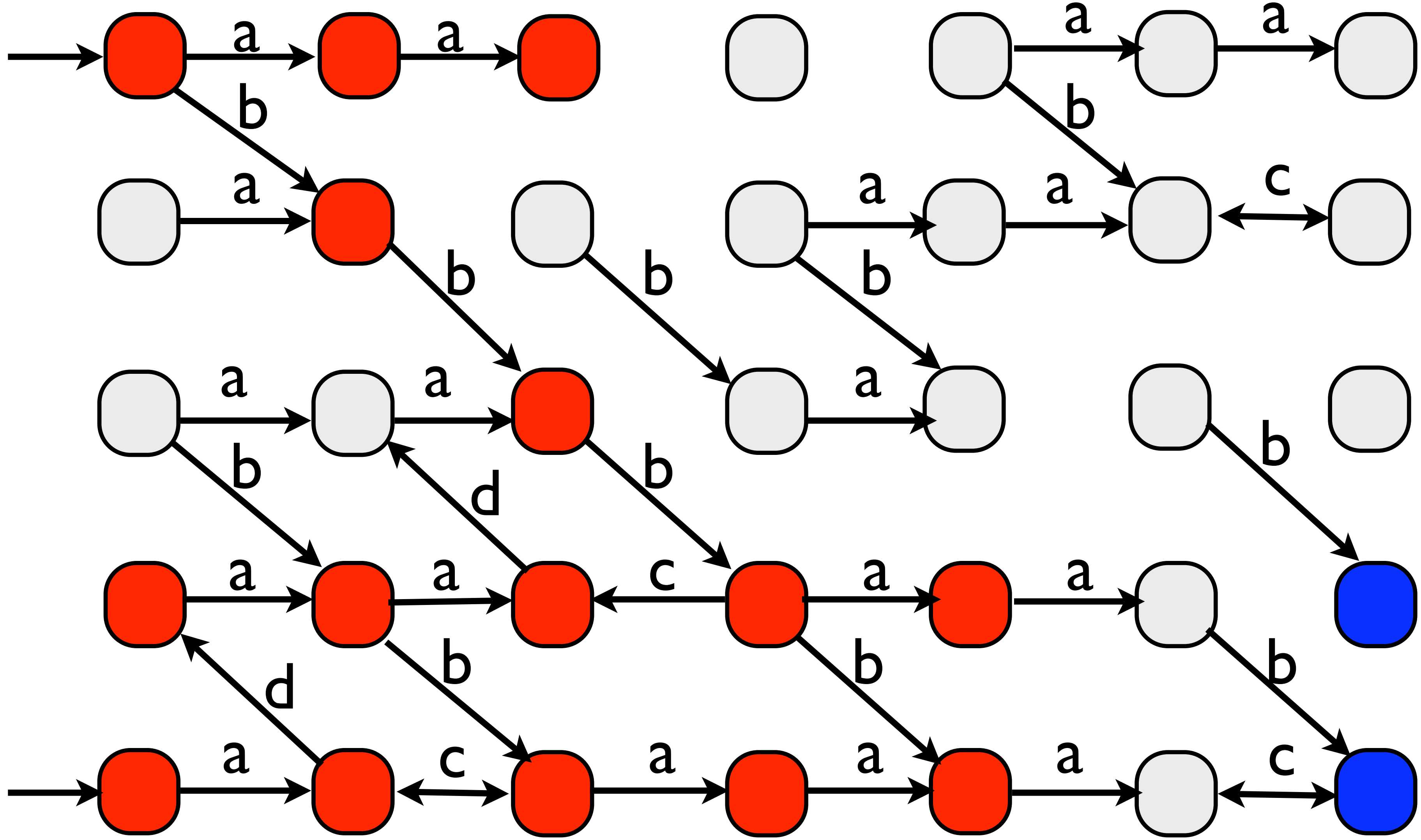
Iterative evaluation of **lfp** ($\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)$)



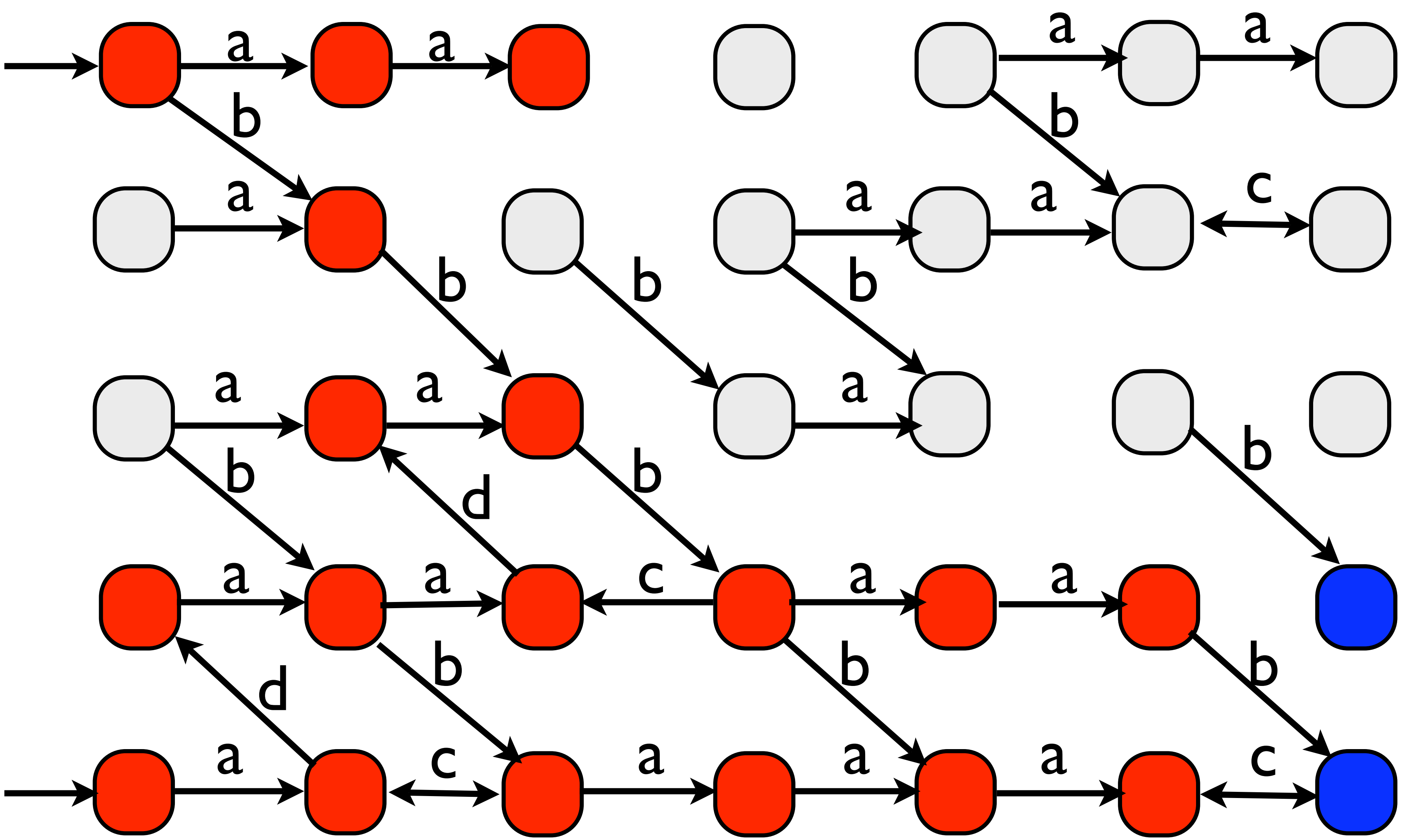
Iterative evaluation of **lfp** ($\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)$)



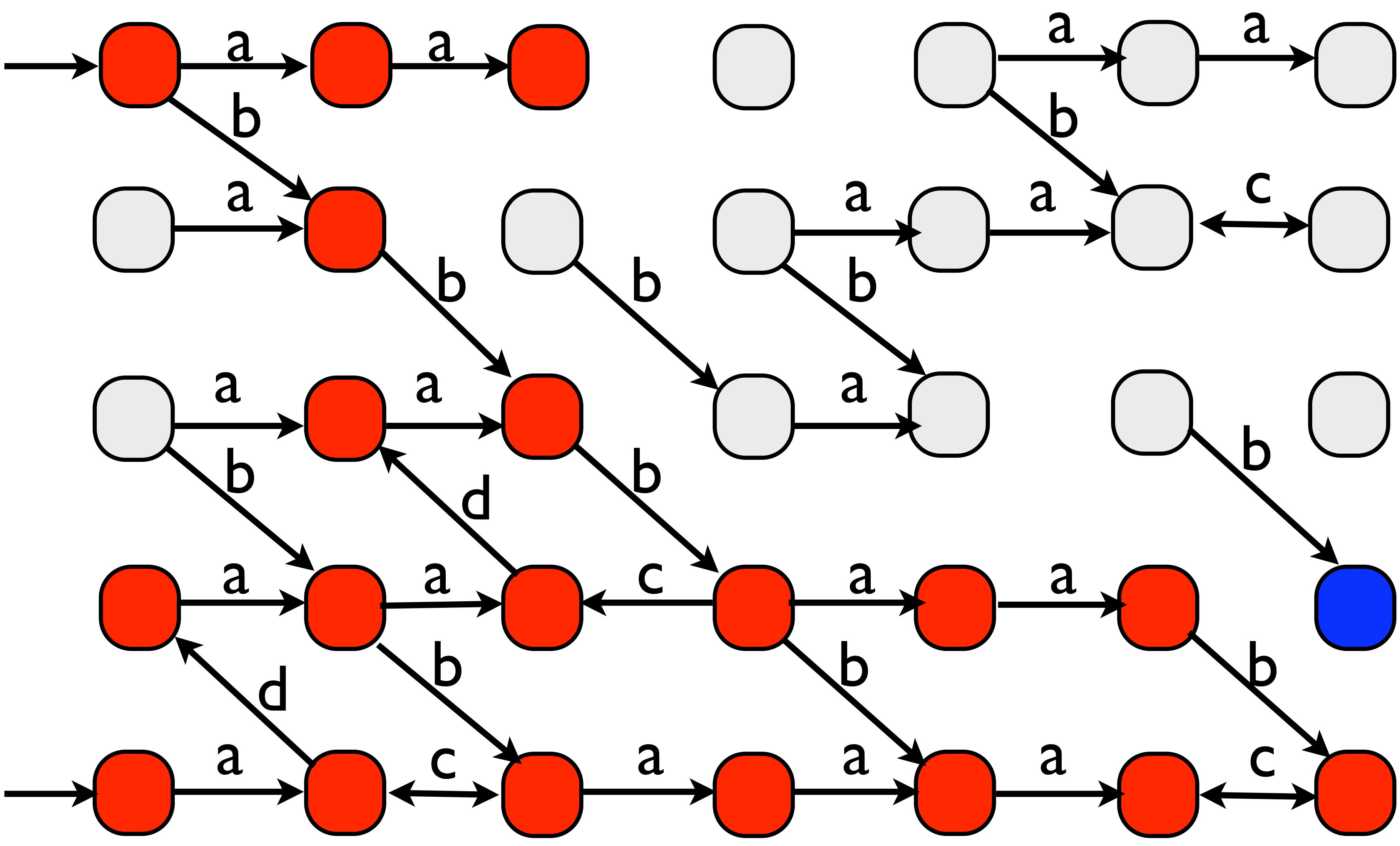
Iterative evaluation of **lfp** ($\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)$)



Iterative evaluation of **lfp** ($\lambda X.\ \mathbf{S_0} \cup \mathbf{POST}(X)$)

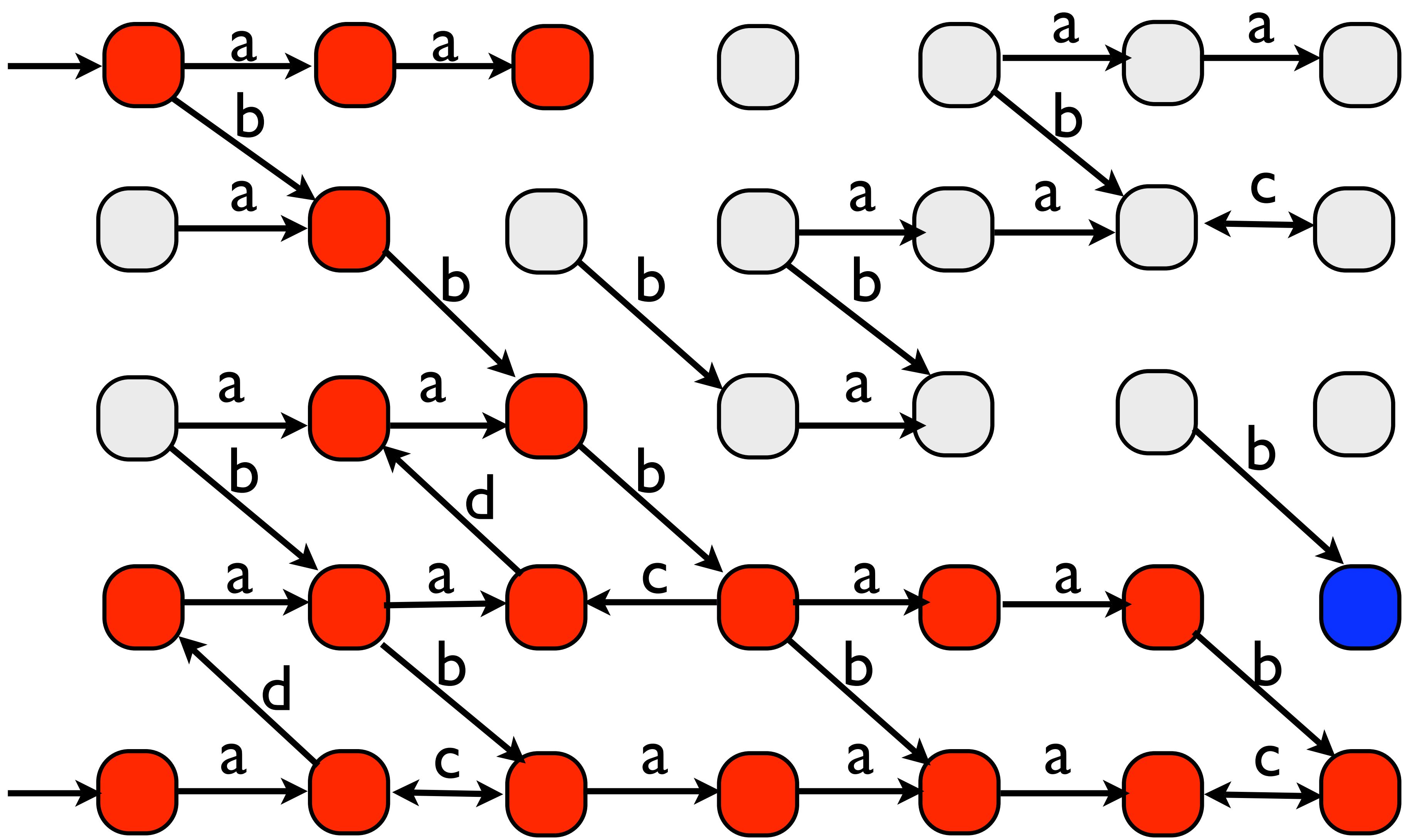


Iterative evaluation of **lfp** ($\lambda X.\, \mathbf{S_0} \cup \mathbf{POST}(X)$)



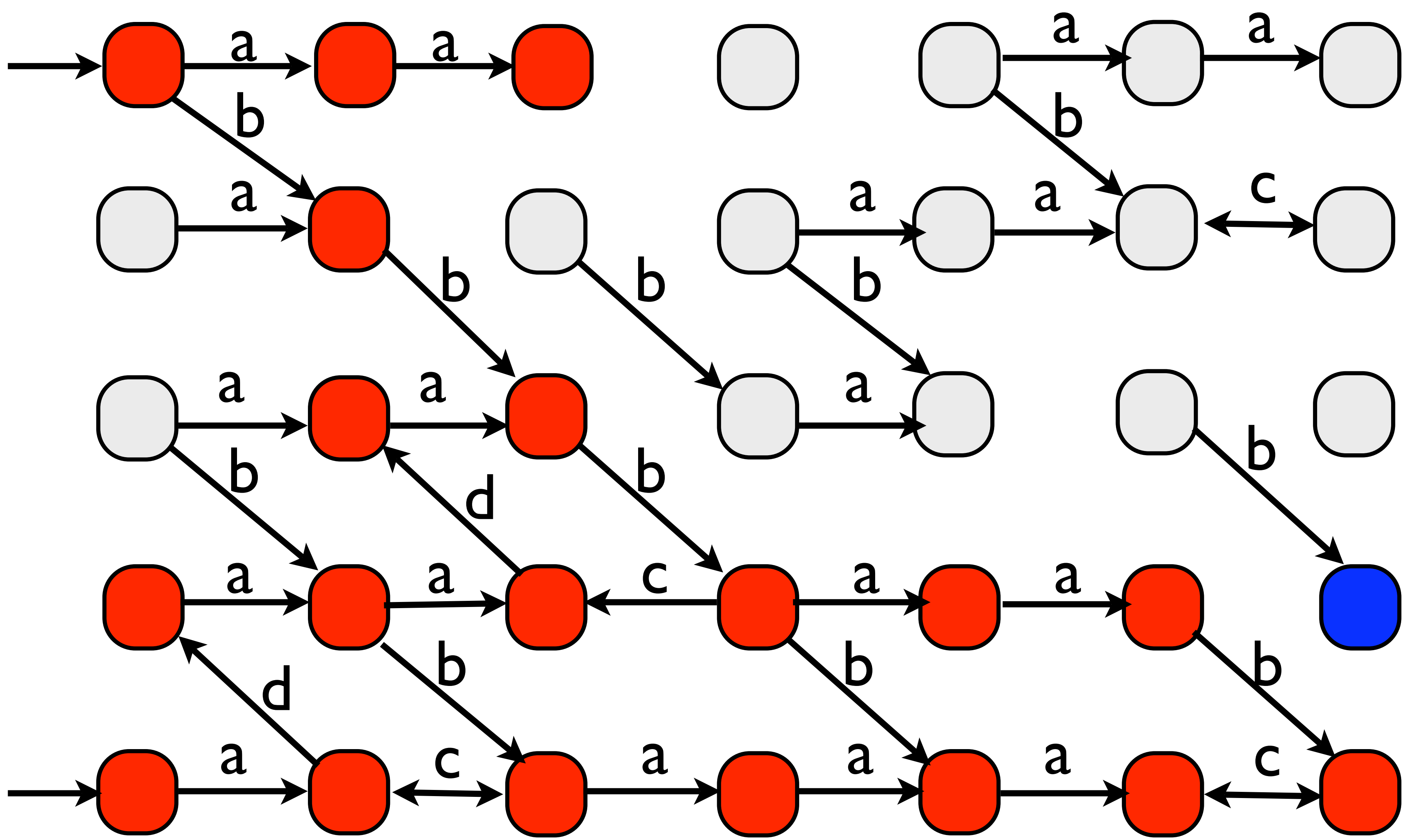
Iterative evaluation of **lfp** (λX. $\mathbf{S_0}$ ∪ **POST**(X))

Fixed point !



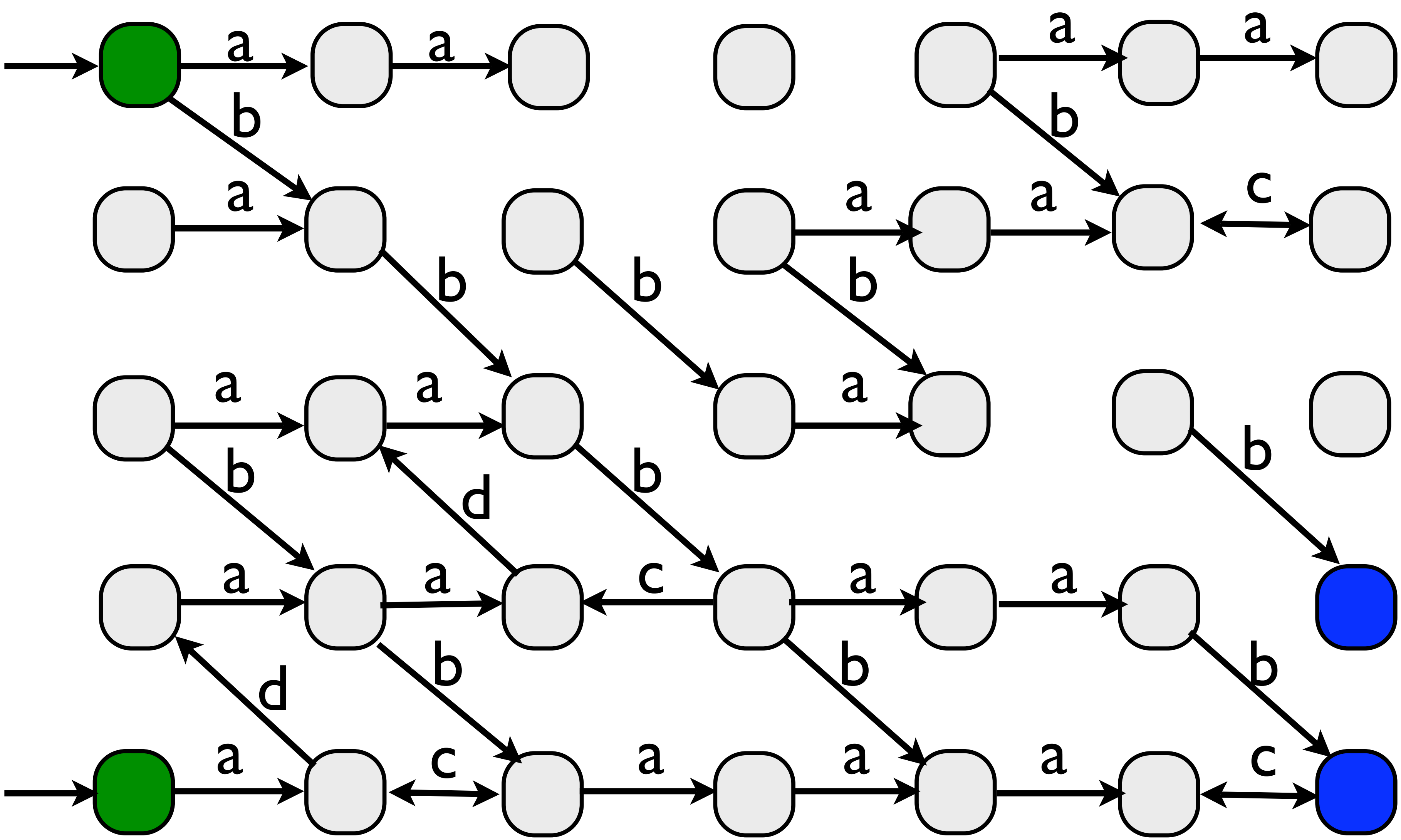
Iterative evaluation of **lfp** $(\lambda X.\, \mathbf{S_0} \cup \mathbf{POST}(X))$

Fixed point ! It intersects Goal ! Positive instance.

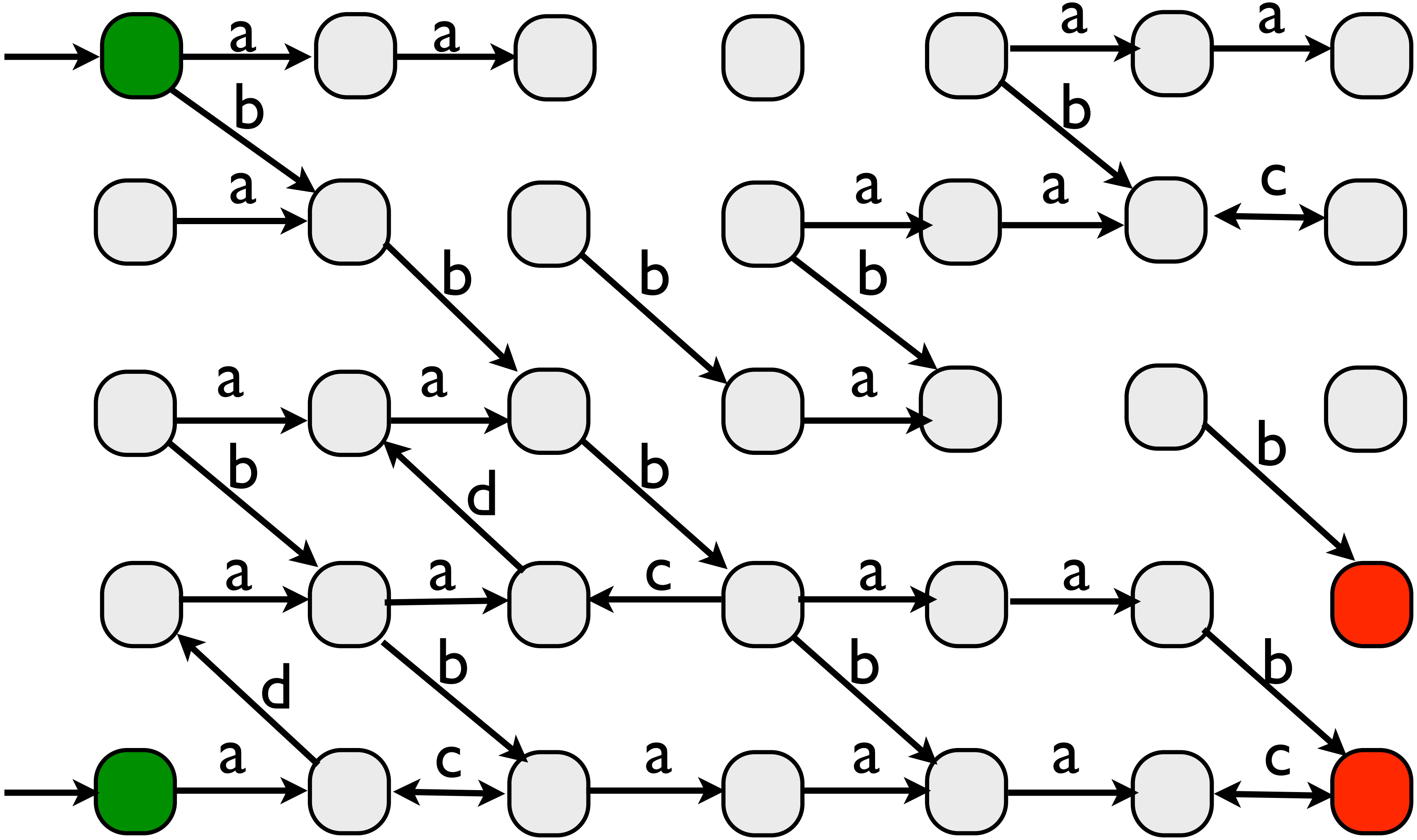


Reachability - Backward algorithm

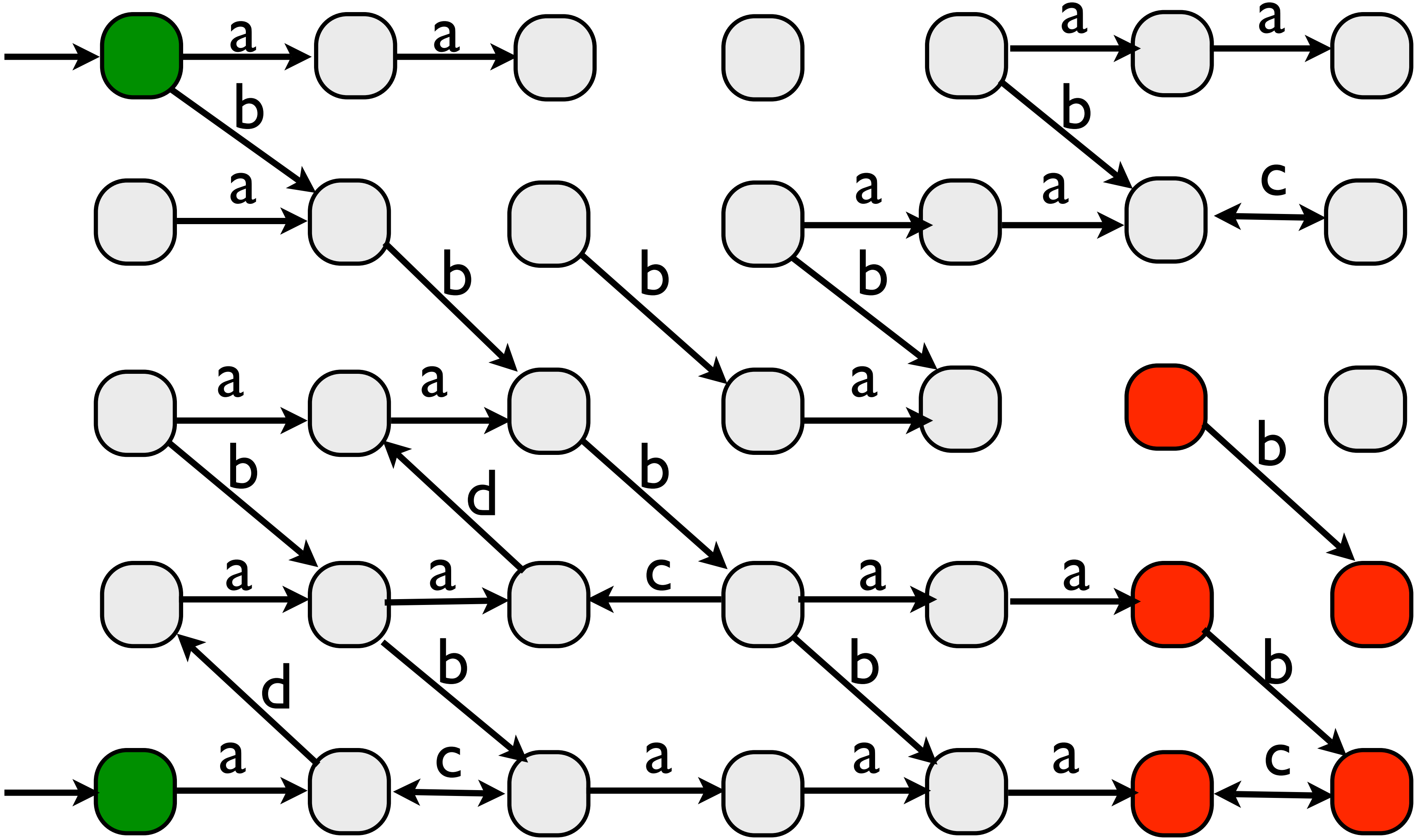
$$\mathbf{lfp}\ (\lambda X.\ \mathbf{Goal} \cup \mathbf{PRE}(X)) \cap \mathbf{S_0} \neq \varnothing$$



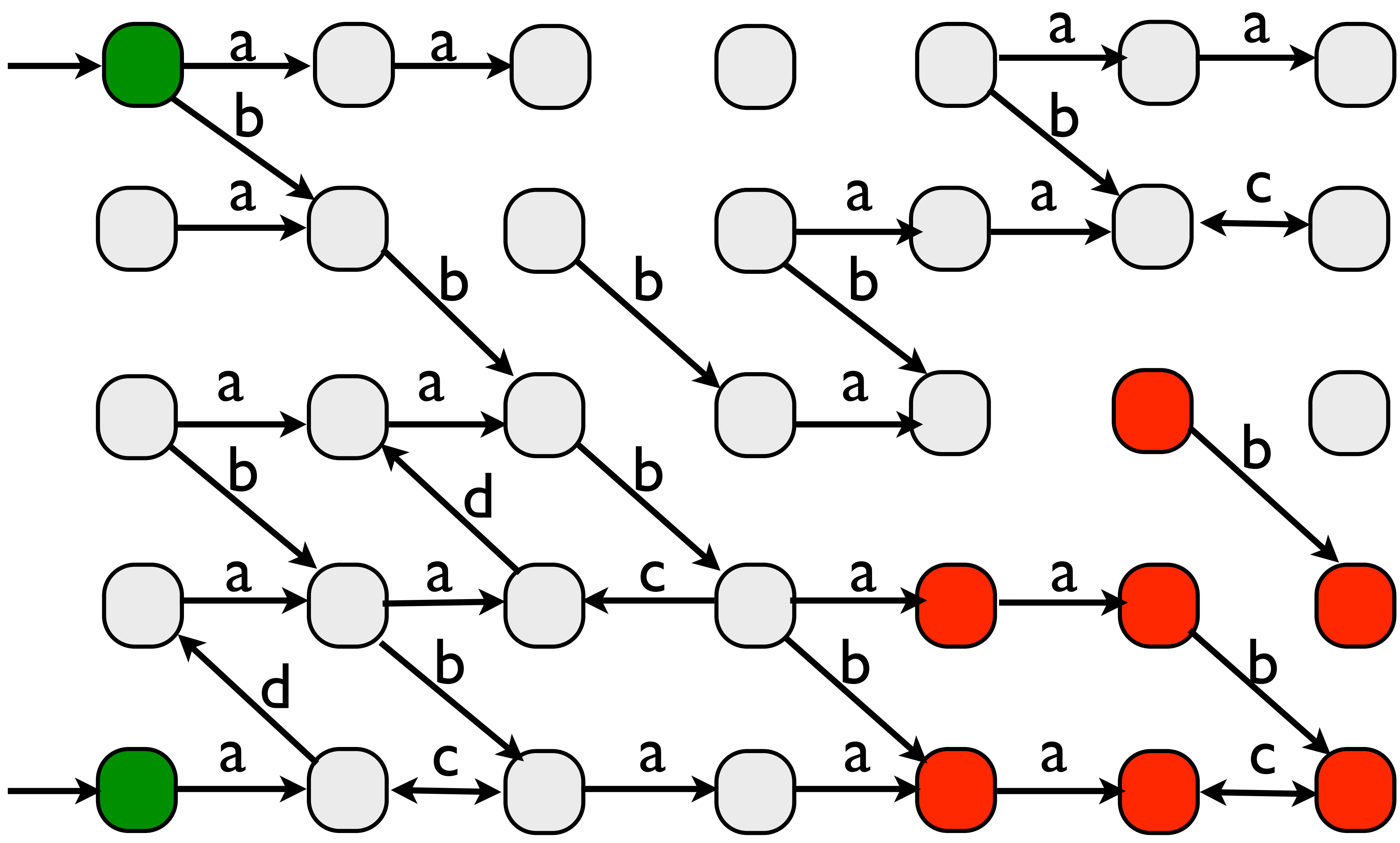
Iterative evaluation of **lfp** (λX. **Goal** $\cup$ **PRE**(X)) $\cap$ $\mathbf{S_0} \neq \emptyset$



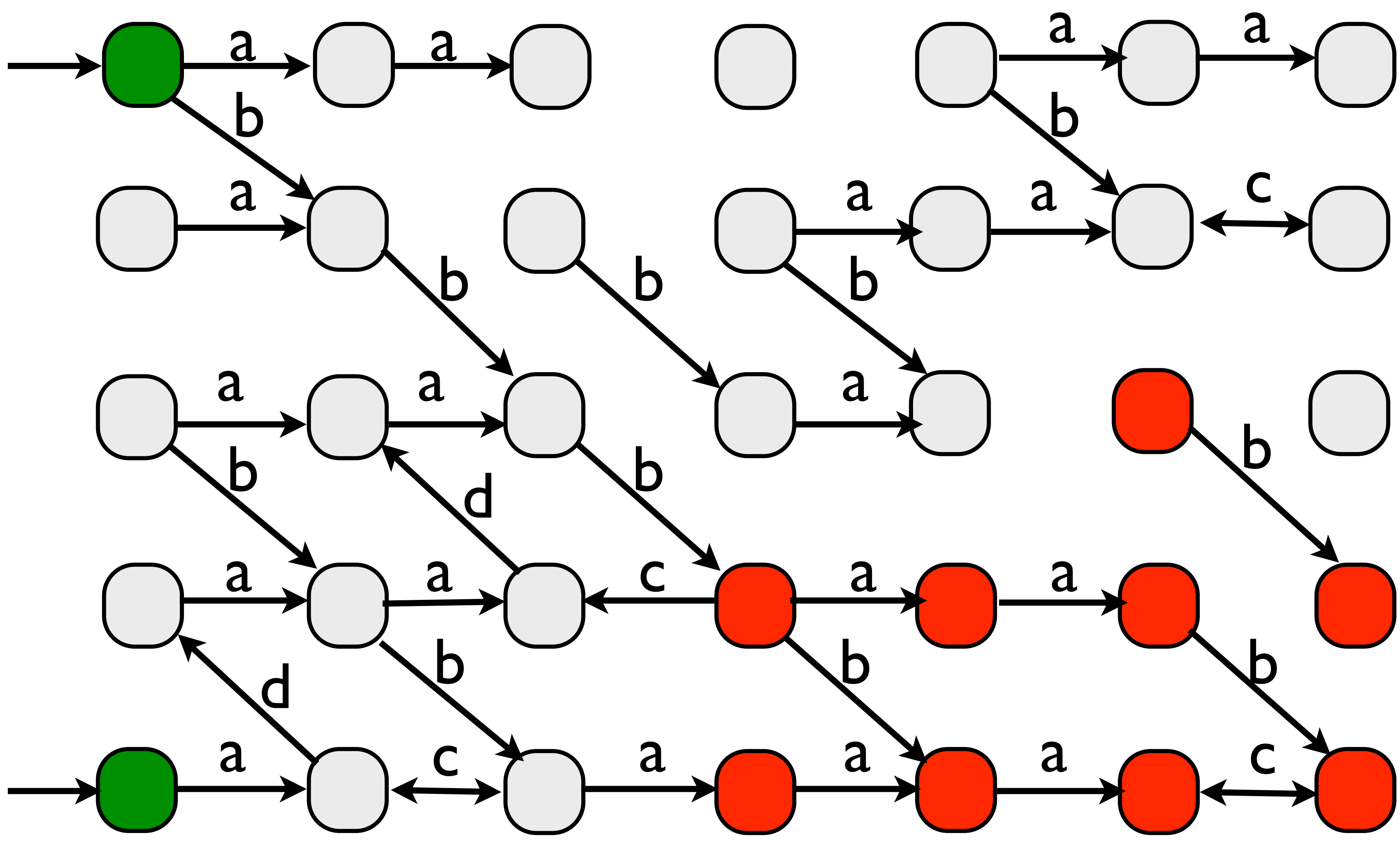
Iterative evaluation of **lfp** (λX. **Goal** ∪ **PRE**(X)) ∩ $\mathbf{S_0} \neq \varnothing$



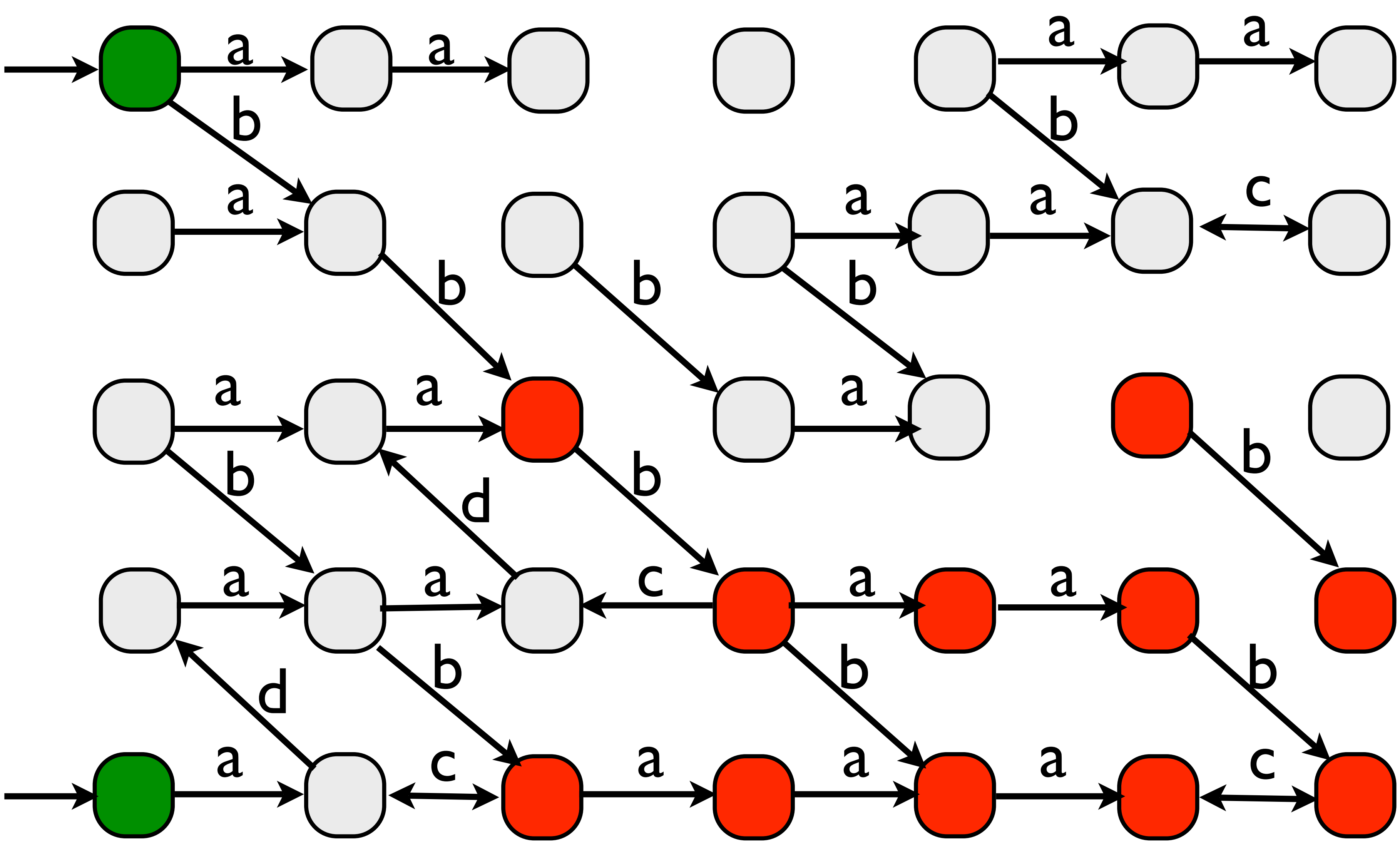
Iterative evaluation of **lfp** (λX. **Goal** ∪ **PRE**(X)) ∩ **S₀**≠∅

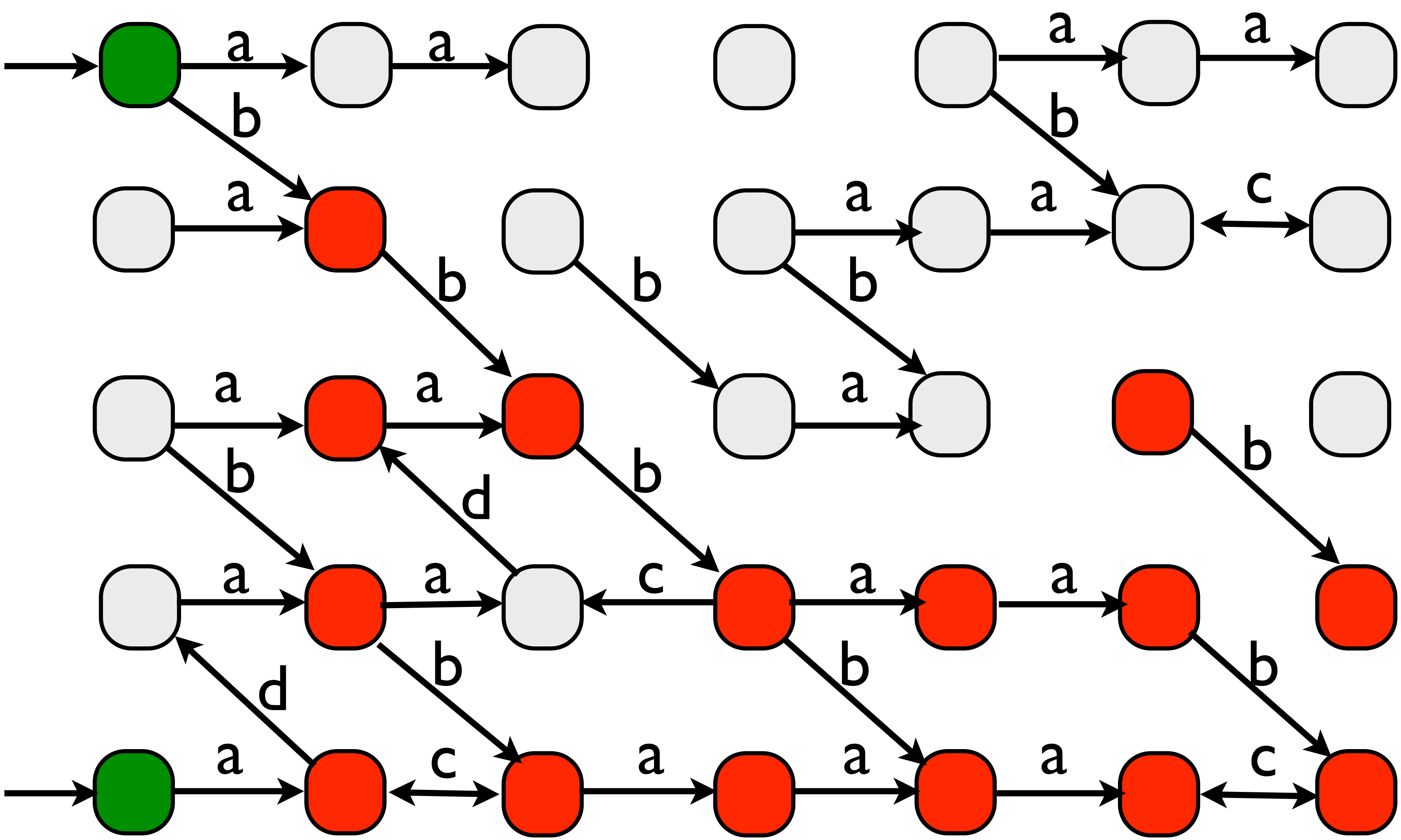


Iterative evaluation of **lfp** (λX. **Goal** ∪ **PRE**(X)) ∩ **S₀**≠∅



Iterative evaluation of **lfp** (λX. **Goal** $\cup$ **PRE**(X)) $\cap$ **S₀**$\neq\varnothing$

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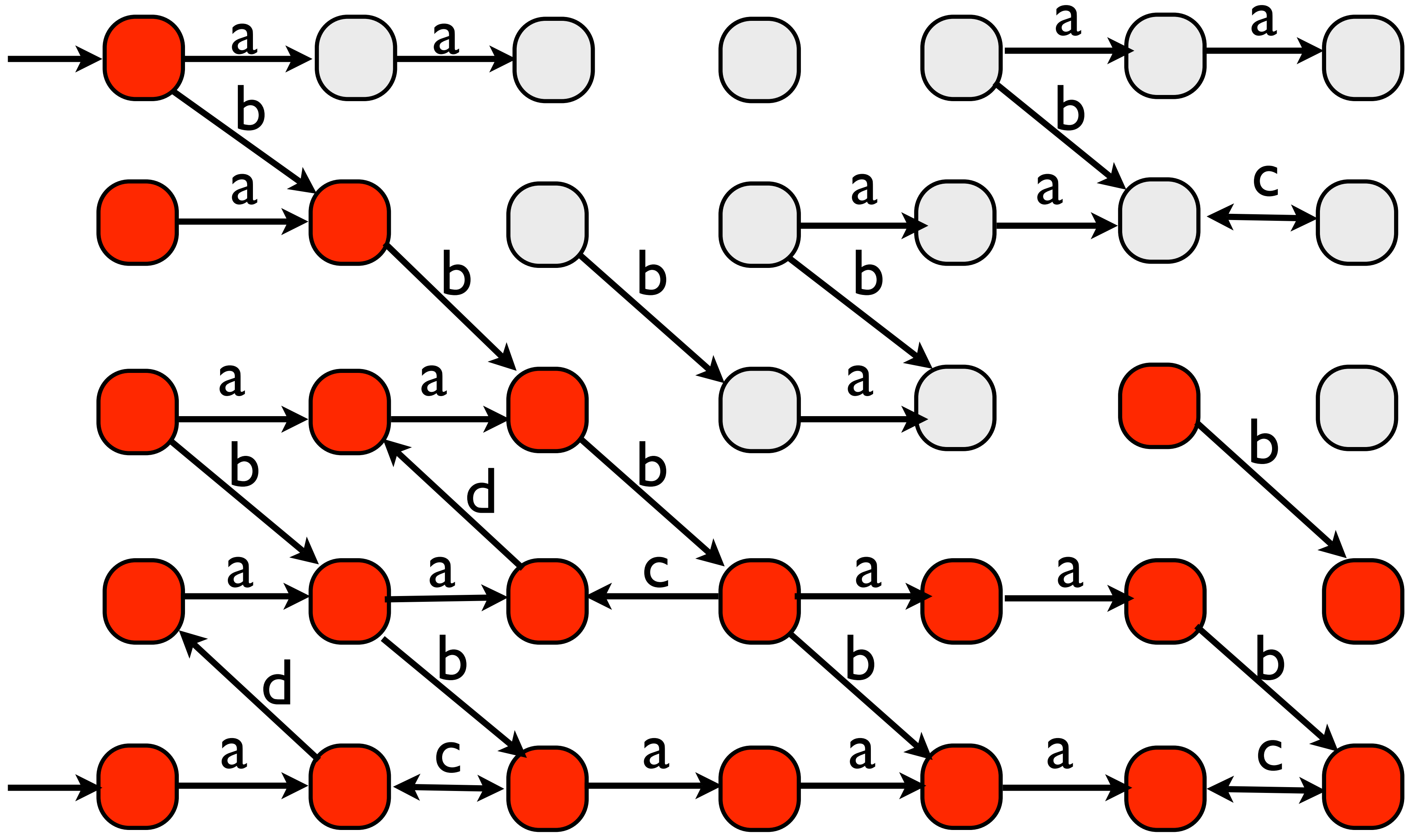
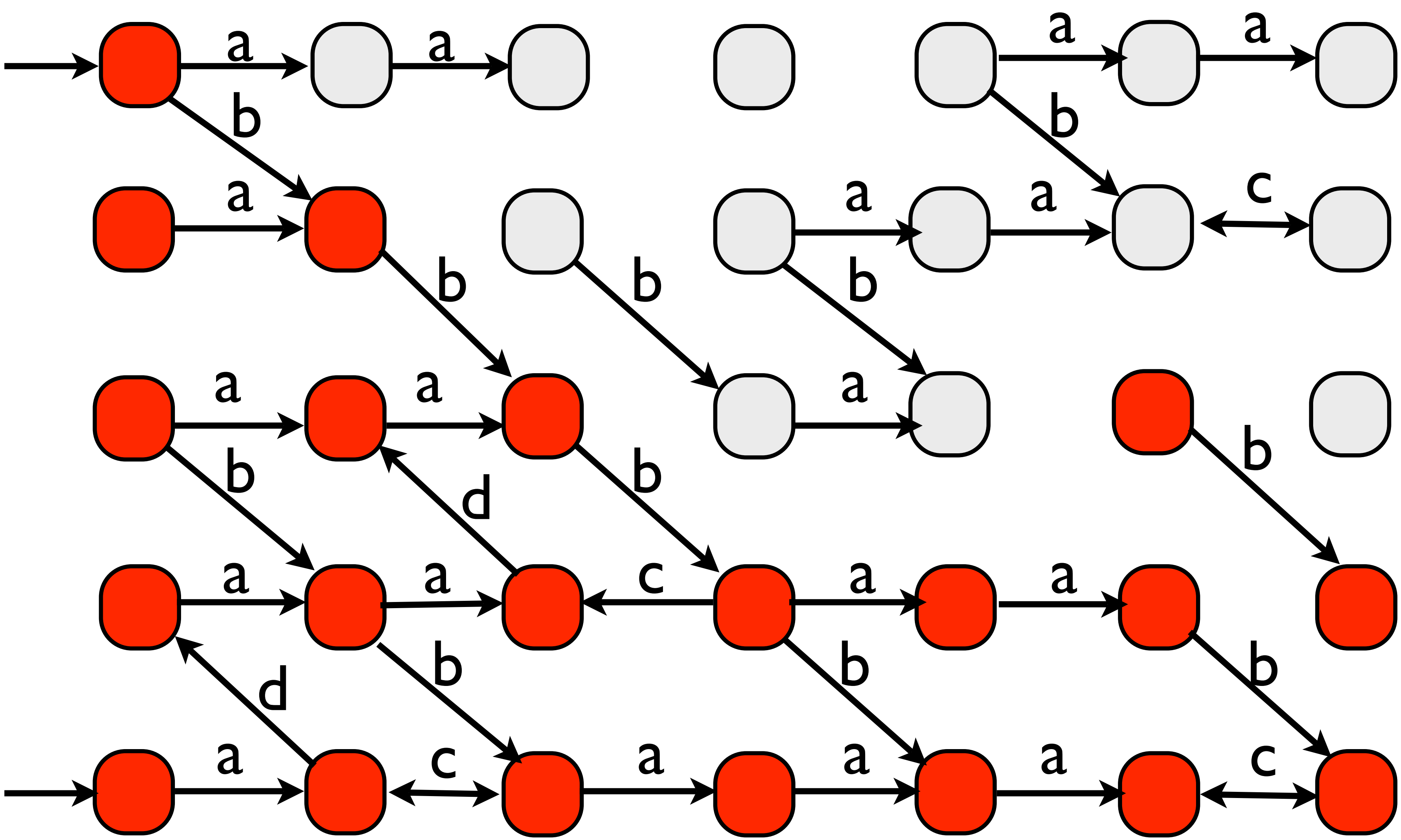
Iterative evaluation of **lfp** (λX. **Goal** ∪ **PRE**(X)) ∩ **S₀**≠∅



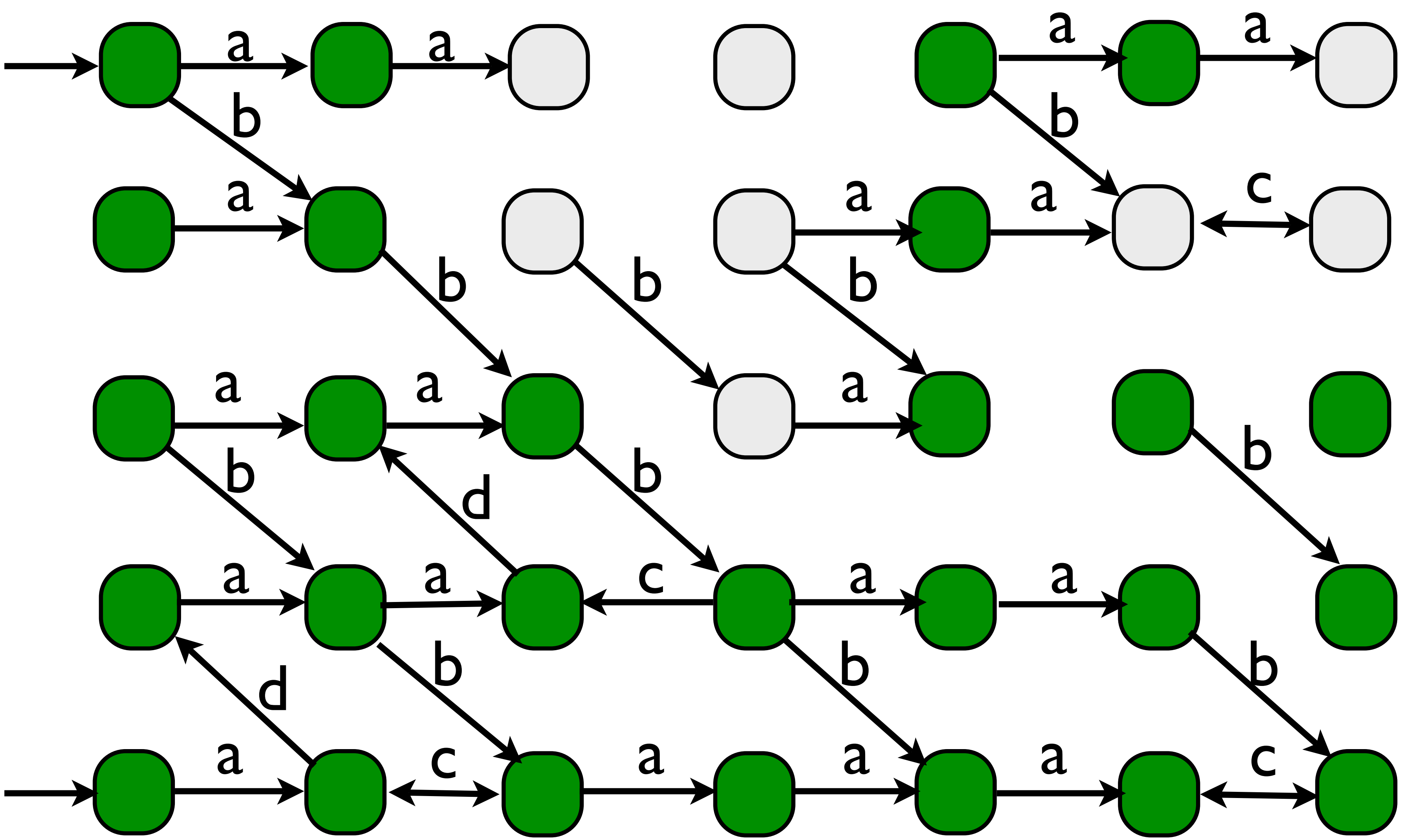
Iterative evaluation of **lfp** (λX. **Goal** $\cup$ **PRE**(X)) $\cap$ $\mathbf{S_0} \neq \varnothing$

Fixed point !   It intersects S_0 ! Positive instance.

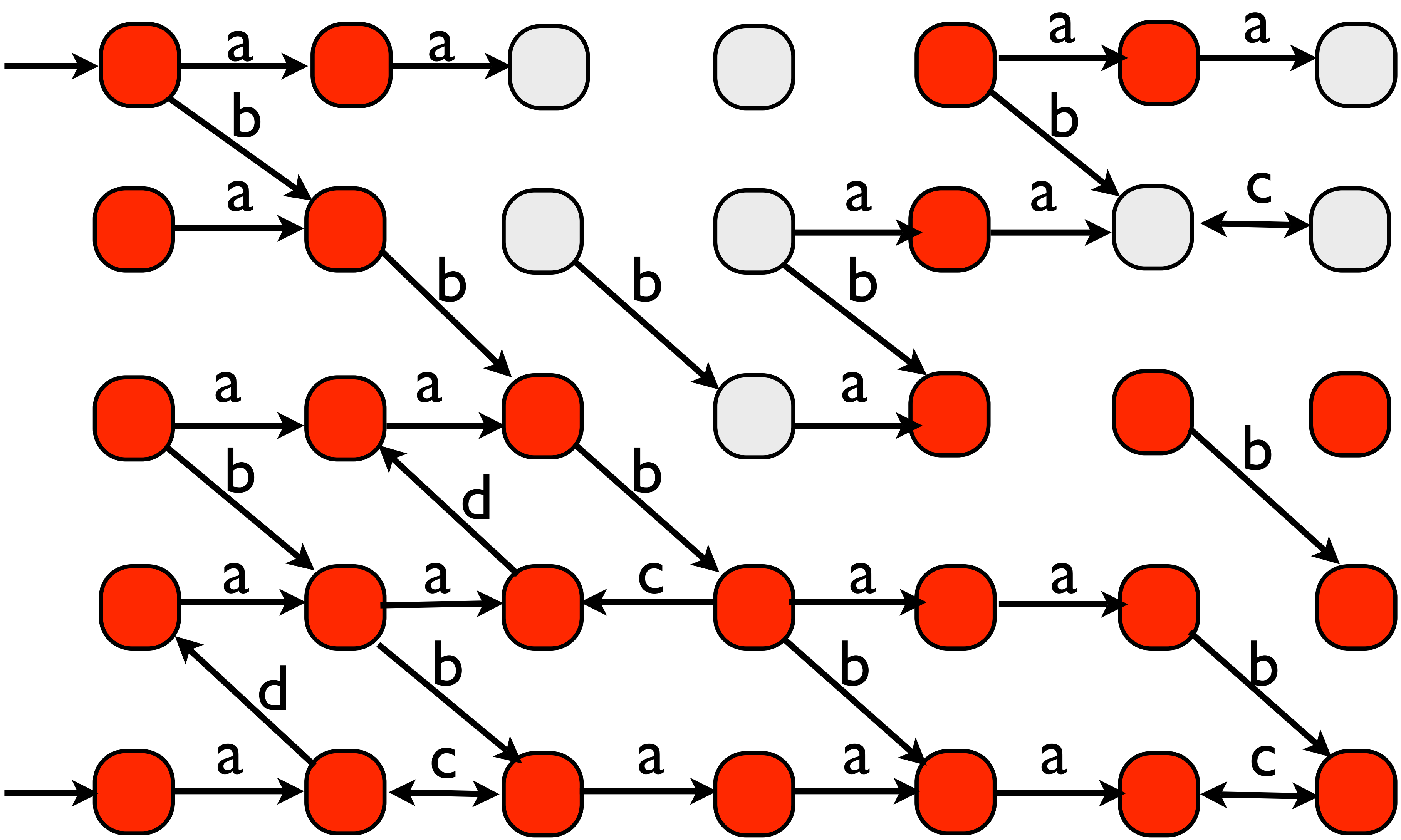


Safety - Backward algorithm

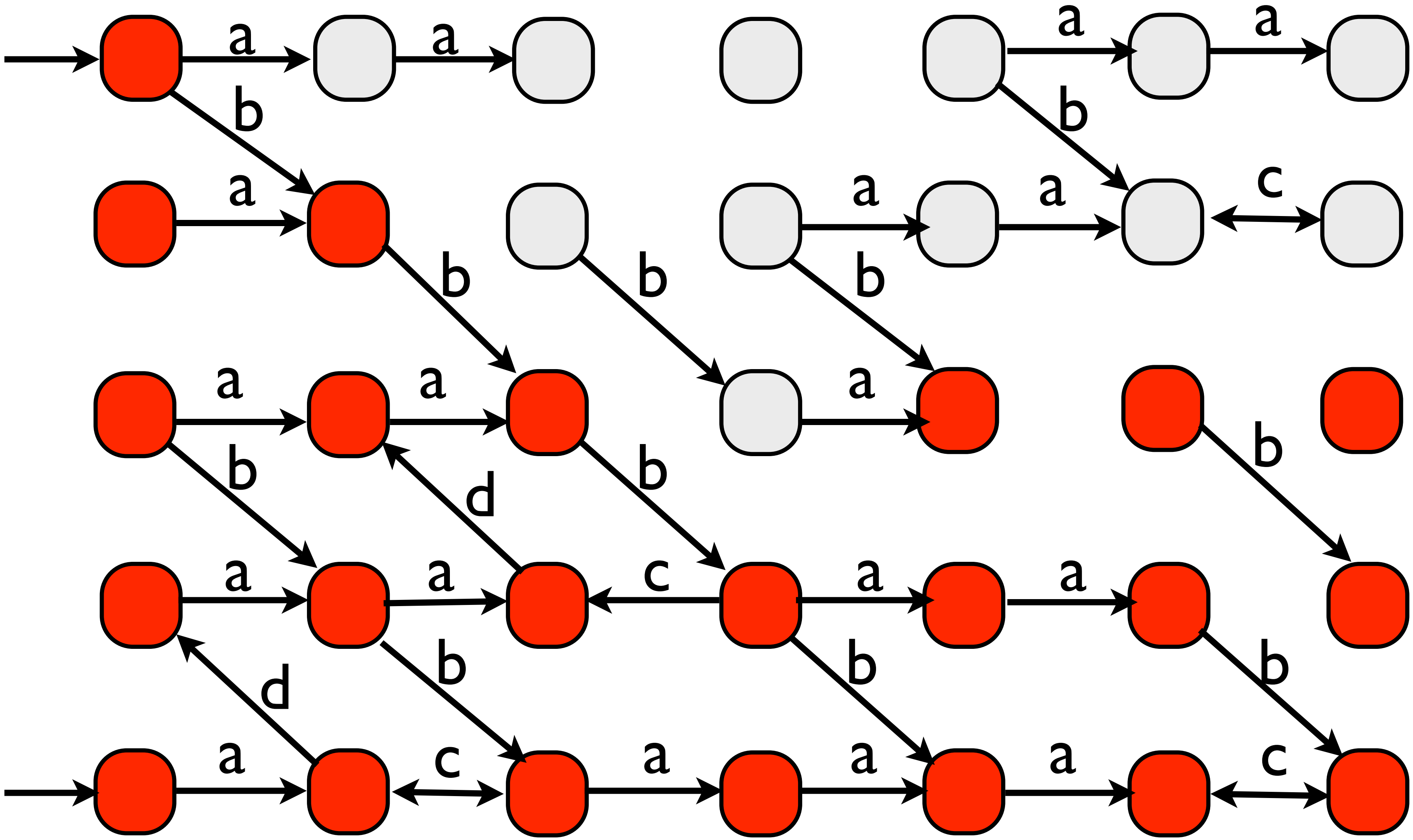
$$\mathbf{S_0} \subseteq \mathbf{gfp}\ (\lambda X.\ \mathbf{Safe} \cap \mathbf{APRE}(X))$$



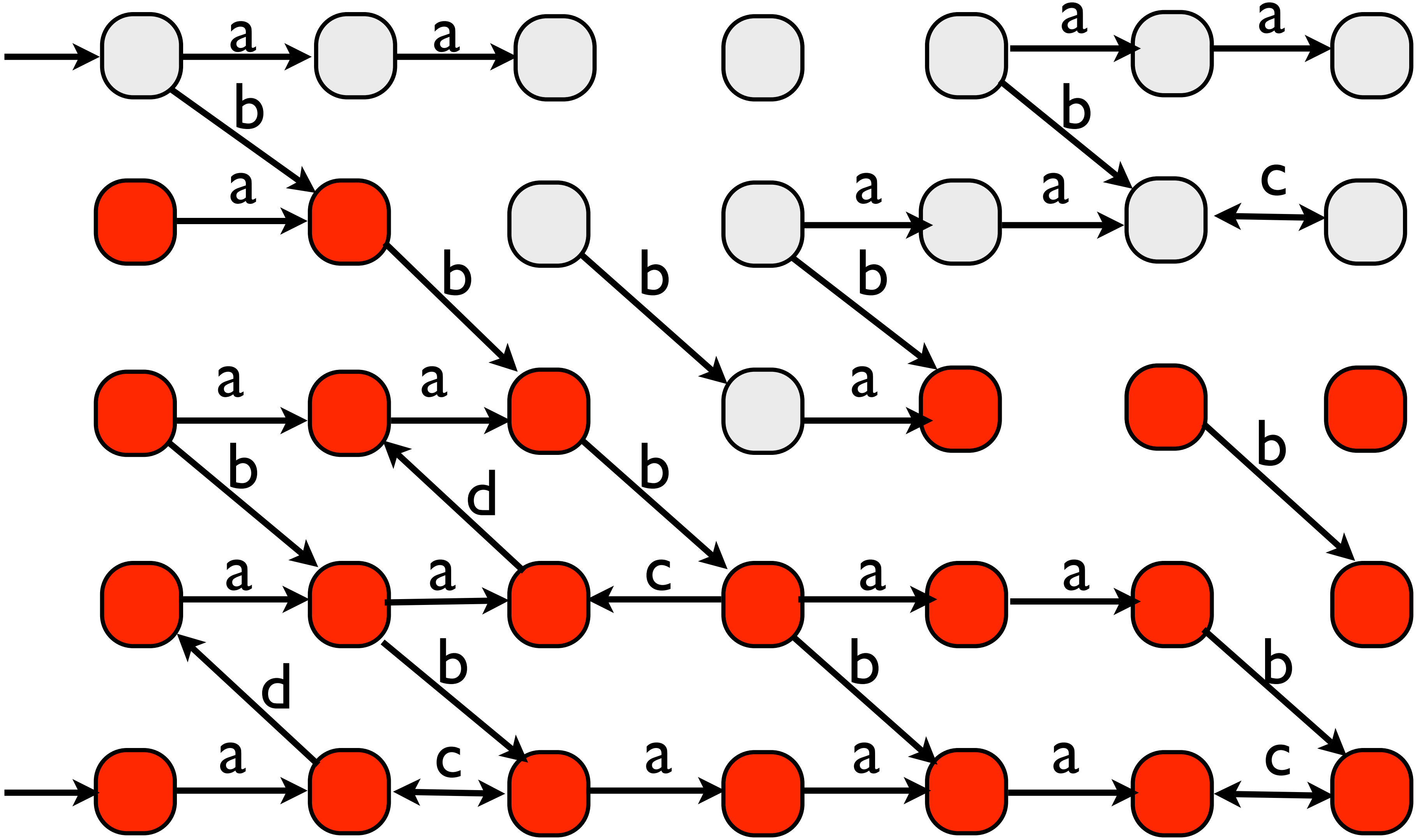
Iterative evaluation of **gfp** (λX. **Safe** ∩ **APRE**(X))



Iterative evaluation of **gfp** (λX. **Safe** ∩ **APRE**(X))

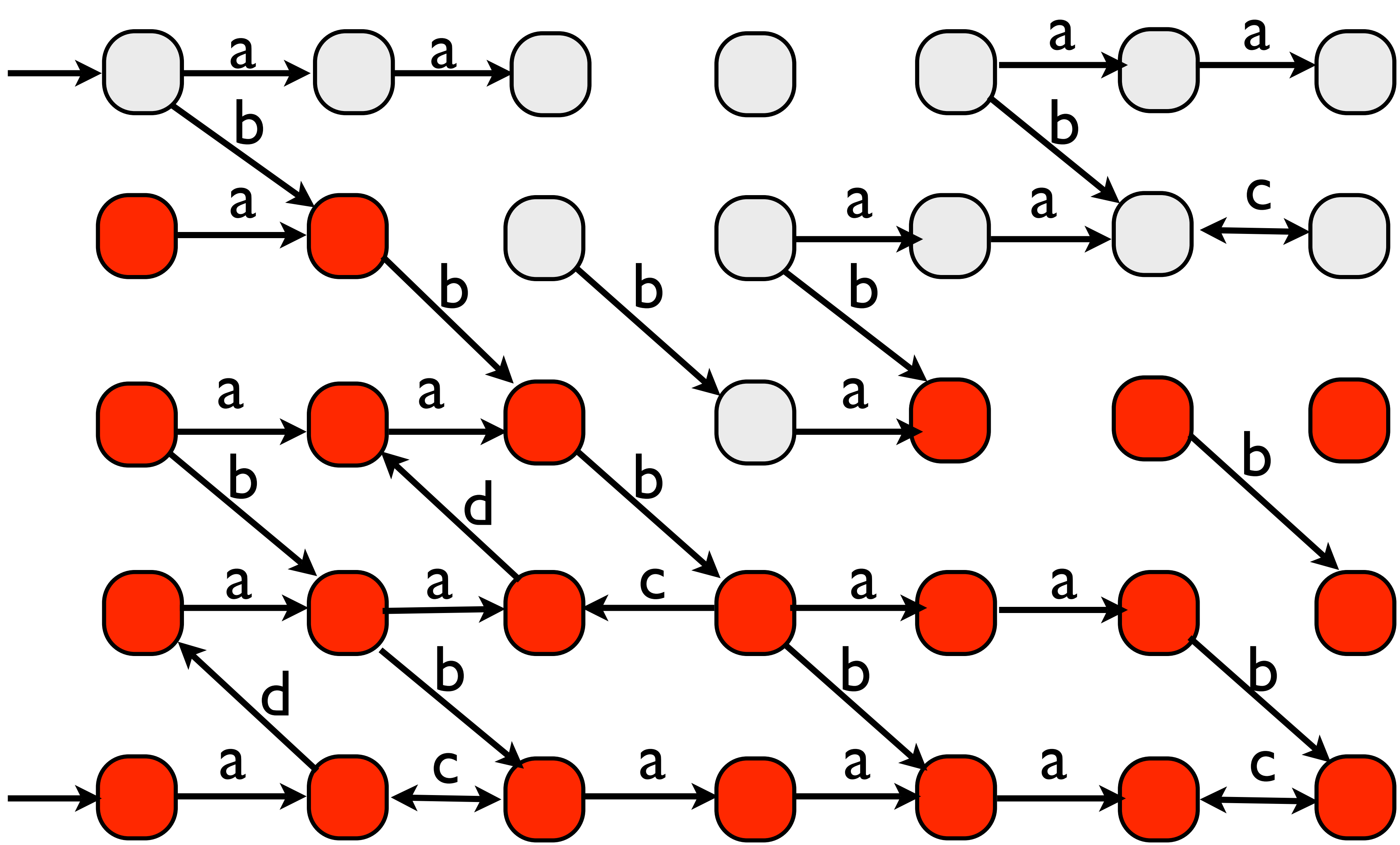


Iterative evaluation of **gfp** (λX. **Safe** ∩ **APRE**(X))



Iterative evaluation of **gfp** (λX. **Safe** ∩ **APRE**(X))

Fixed point ! Negative instance as $S_0 \not\subseteq$ **gfp** (λX. **Safe** ∩ **APRE**(X)).



Fixed points for Büchi objectives

- Let consider an instance of the **Büchi verification problem** given by the LTS $L=(S,S_0,\Sigma,T,C,\lambda)$, a set of states **Goal** $\subseteq S$;

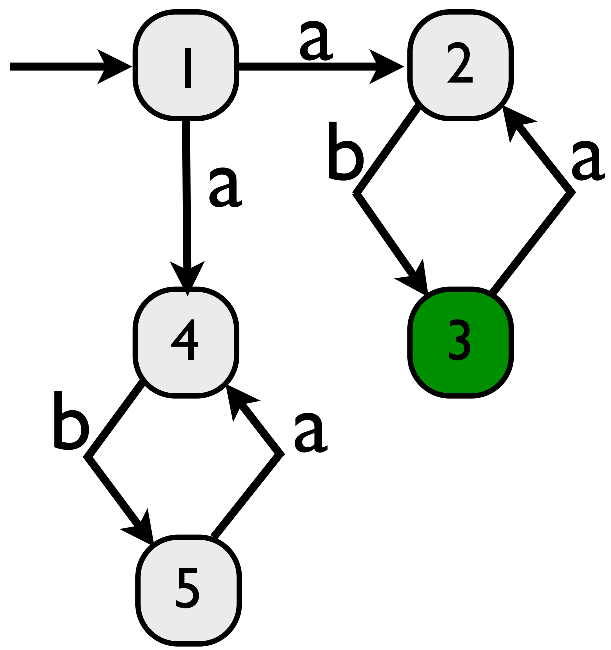
- **Goal** is reachable infinitely often from an initial states in L

  iff

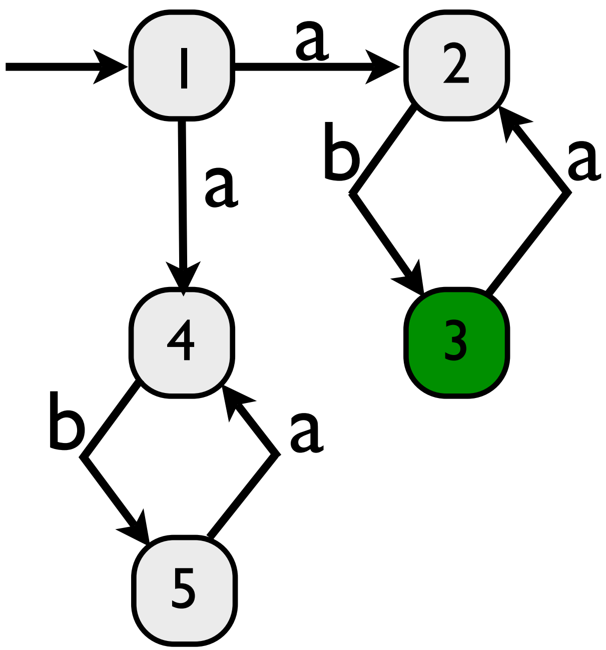
  $\mathbf{gfp}(\lambda Y.\, \mathbf{lfp}(\lambda X.\, \mathbf{PRE}(X) \cup (\, \mathbf{Goal} \cap \mathbf{PRE}(Y)))) \cap \mathbf{S_0} \neq \emptyset$

  this is a backward algorithm

Fixed points for Büchi objectives

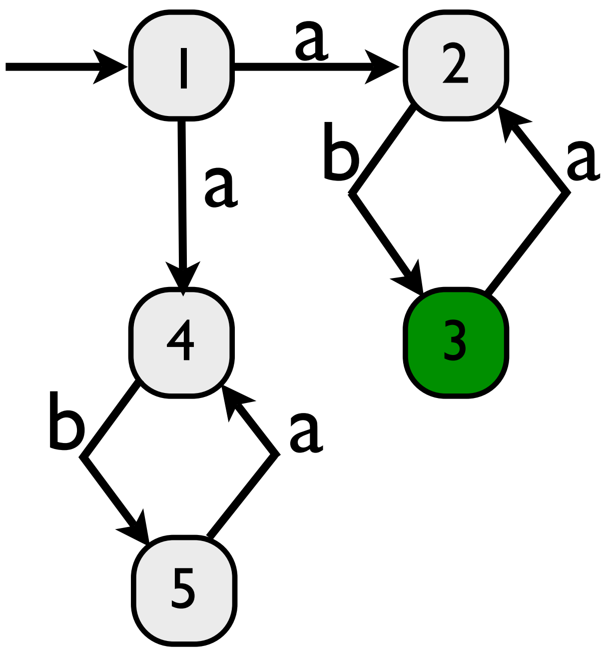


Fixed points for Büchi objectives



We want to check if {3} can be reached infinitely often from the initial state.

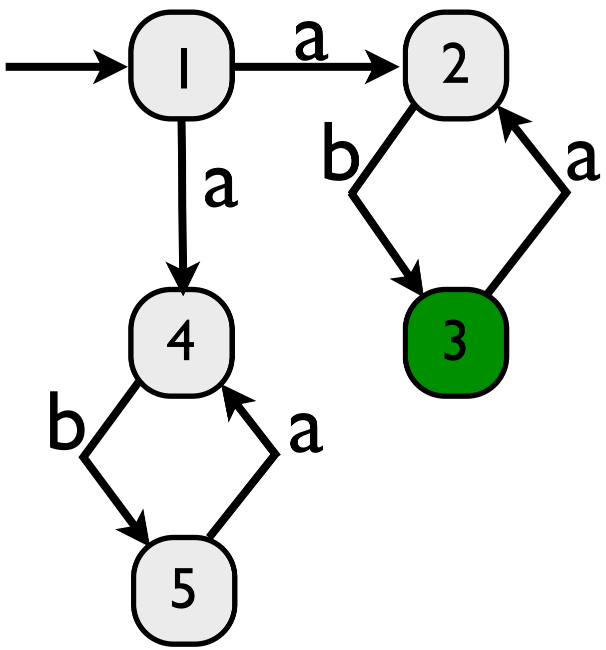
Fixed points for Büchi objectives



We want to check if {3} can be reached infinitely often from the initial state.

For that we evaluate the fixed point expression
$\mathbf{gfp}(\lambda Y.\, \mathbf{lfp}(\lambda X.\, \mathbf{PRE}(X) \cup (\mathbf{Goal} \cap \mathbf{PRE}(Y))))$

Fixed points for Büchi objectives



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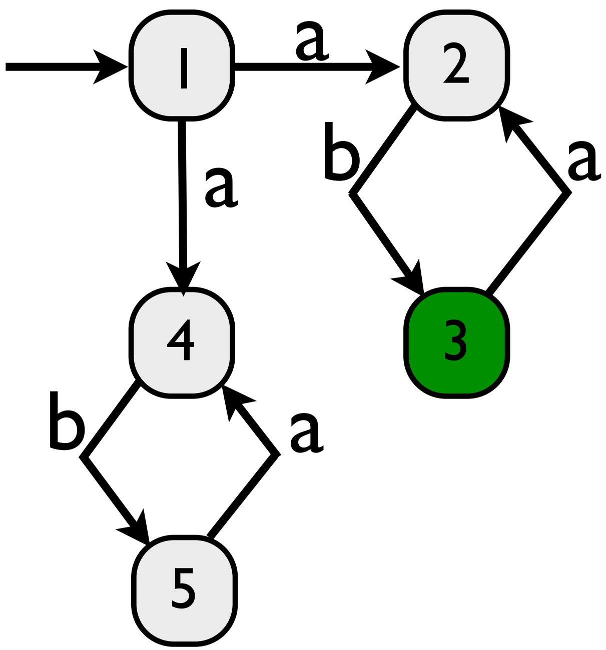

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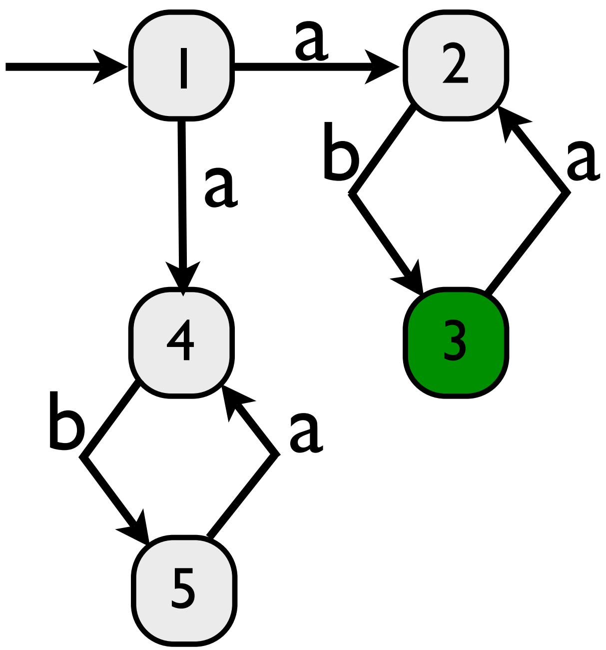

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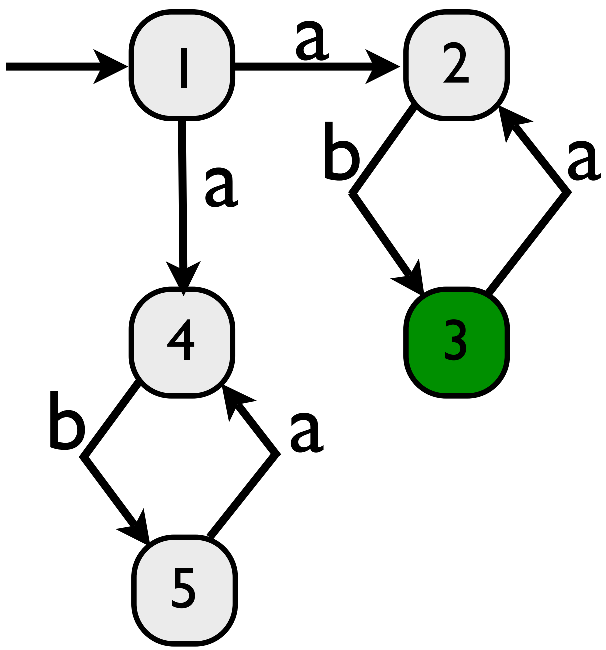

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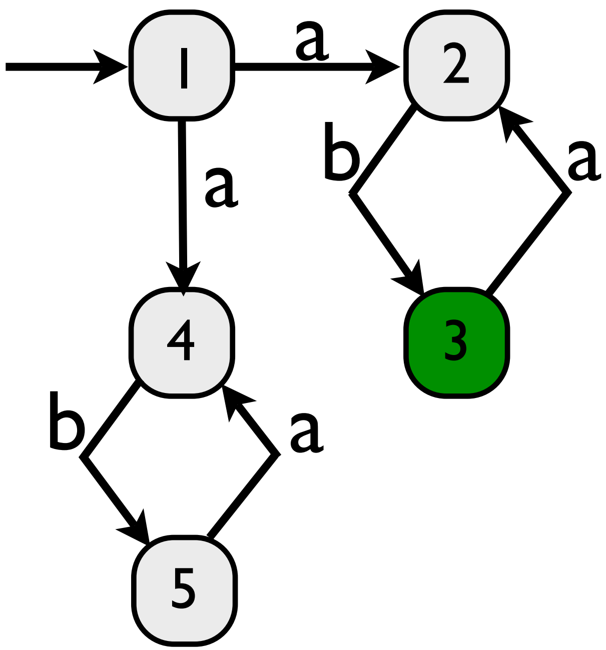
Fixed points for Büchi objectives



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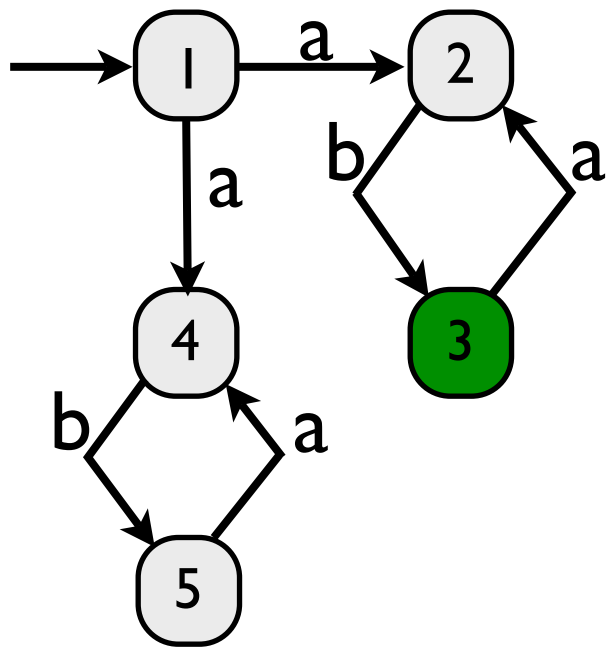
Fixed point !

Fixed points for Büchi objectives



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Fixed point !

As $S_0 \cap Y_2 \neq \varnothing$, the Büchi property is verified by the LTS L

Trace pre-orders, Trace equivalence, Simulations, Bisimulations and quotients

Traces of a LTS

- Let $(S,S_0,\Sigma,T,C,\lambda)$ be a LTS. Let $s_0\sigma_0 s_1\sigma_1 s_2\sigma_2\ldots\sigma_{n-1}s_n\ldots$ be such that (1) $s_0 \in S_0$, (2) $\forall i \cdot 0 \leq i \cdot T(s_i,\sigma_i,s_{i+1})$, the sequence

$$\lambda(s_0)\sigma_0\, \lambda(s_1)\sigma_1\lambda(s_2)\sigma_2\ldots\sigma_{n-1}\lambda(s_n)\ldots$$

  is called a **trace** of the LTS.

- Note that two **different** paths in the LTS may generate the **same** trace.

- The color of a state is meant to model the **important properties** of that state. So the notion of trace allows us to **concentrate** on the important properties of the system.

Traces of a LTS

- We note **Traces**(L) the set of traces generated by the LTS L.
- Two LTS L_1 and L_2 are **trace equivalent** if

  Traces(L_1)=**Traces**(L_2)

- **Trace equivalence and verification**.
  If we have two LTS L_1 and L_2 such that **Traces**(L_1)=**Traces**(L_2), and L_2 is (much) **smaller** than L_1, it may be very advantageous to do verification on L_2 instead on L_1. As we will see L_1 may be infinite while L_2 is finite. We will illustrate that with TA.
- Unfortunately, minimizing a system using the notion of trace equivalence is **costly** computationally. We will introduce now **stronger notions of equivalence** than are easier to compute.

Simulation relations

Simulation relations

- Given a LTS $(S,S_0,\Sigma,T,C,\lambda)$, a **simulation relation** is a relation $\mathbf{R}\subseteq S\times S$ such that

  for all $(s_1,s_2) \in \mathbf{R}$:
  (1) $s_1\in S_0$ iff $s_2\in S_0$
  (2) $\lambda(s_1)=\lambda(s_2)$
  (3) $\forall\sigma\in\Sigma\bullet\forall s_3\in S\colon\ T(s_1,\sigma,s_3)\Longrightarrow\exists s_4\in S\bullet T(s_2,\sigma,s_4)\wedge(s_3,s_4)\in\mathbf{R}$

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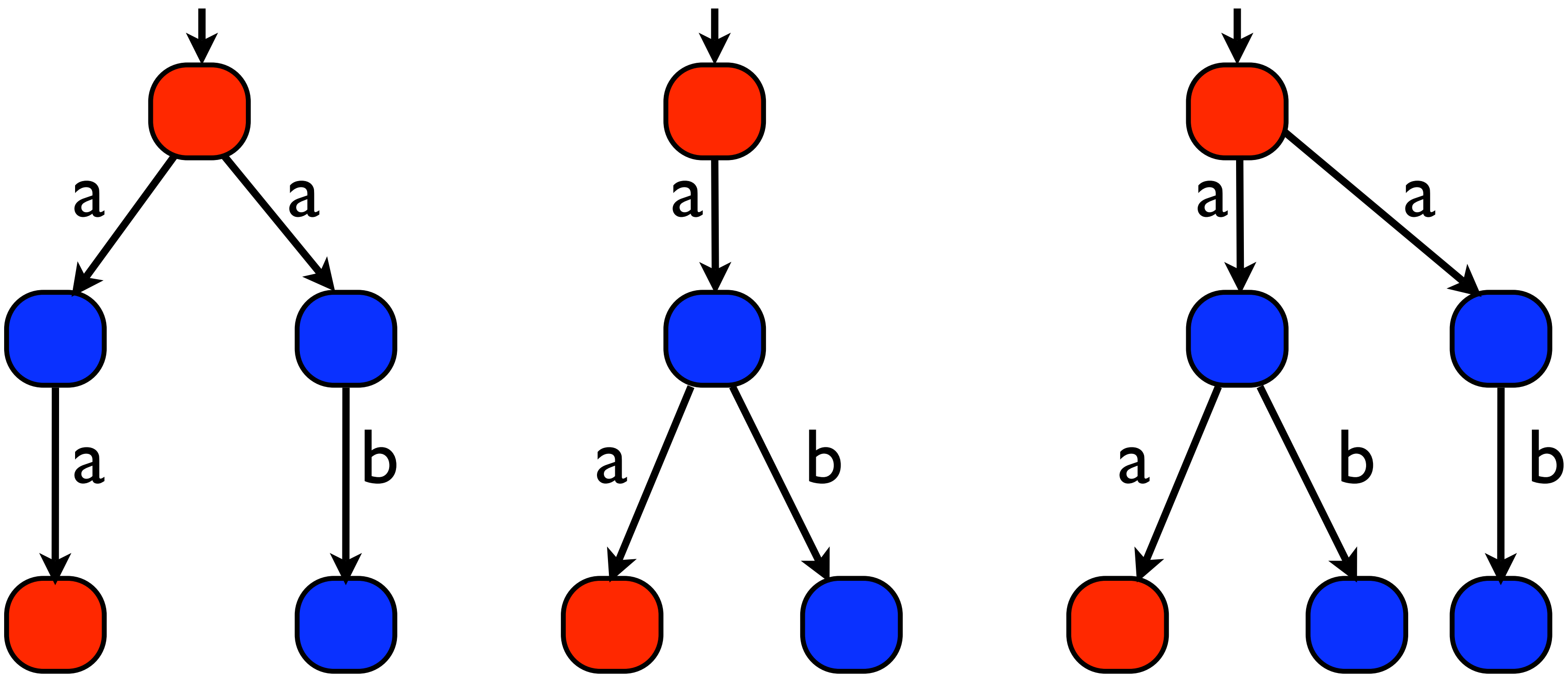
- When $(s_1,s_2) \in \mathbf{R}$, we say that **s_1 is simulated by s_2**.

Simulation relations



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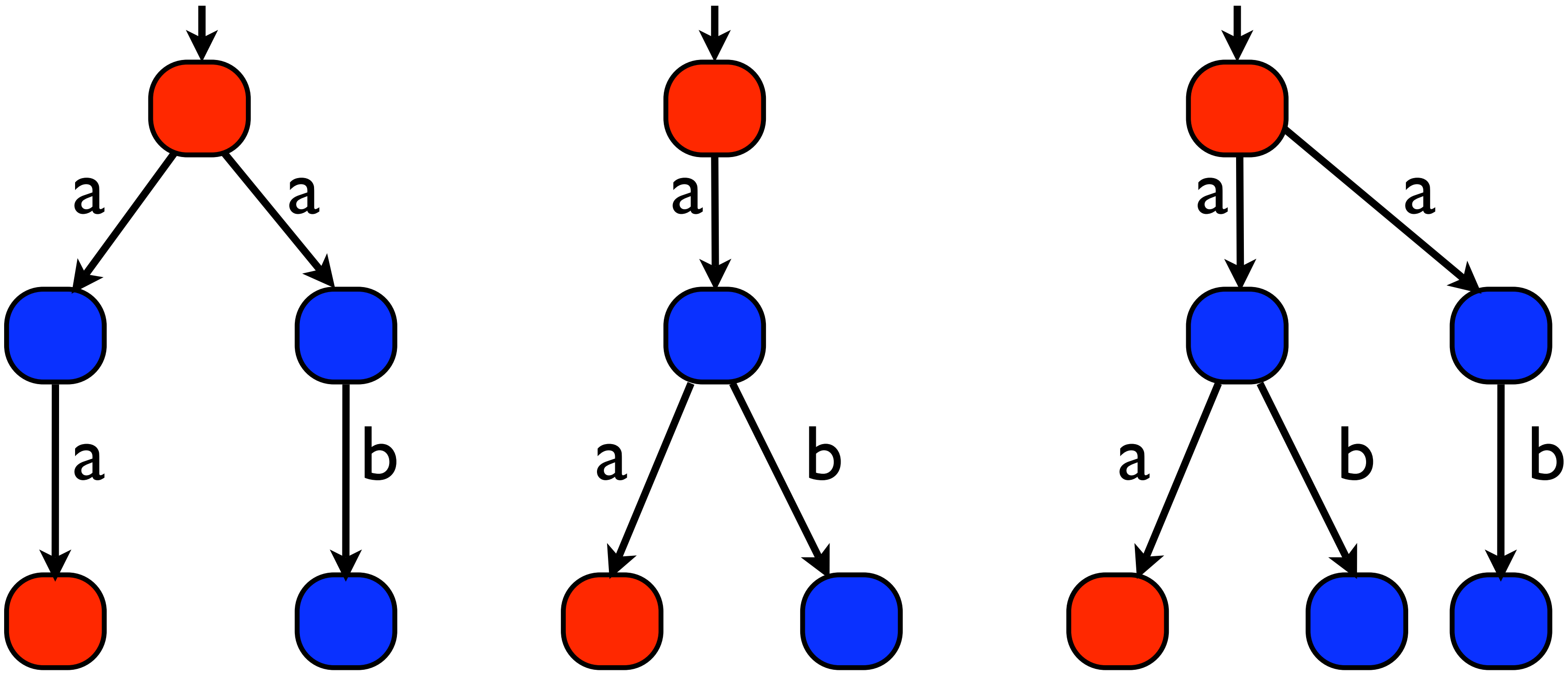
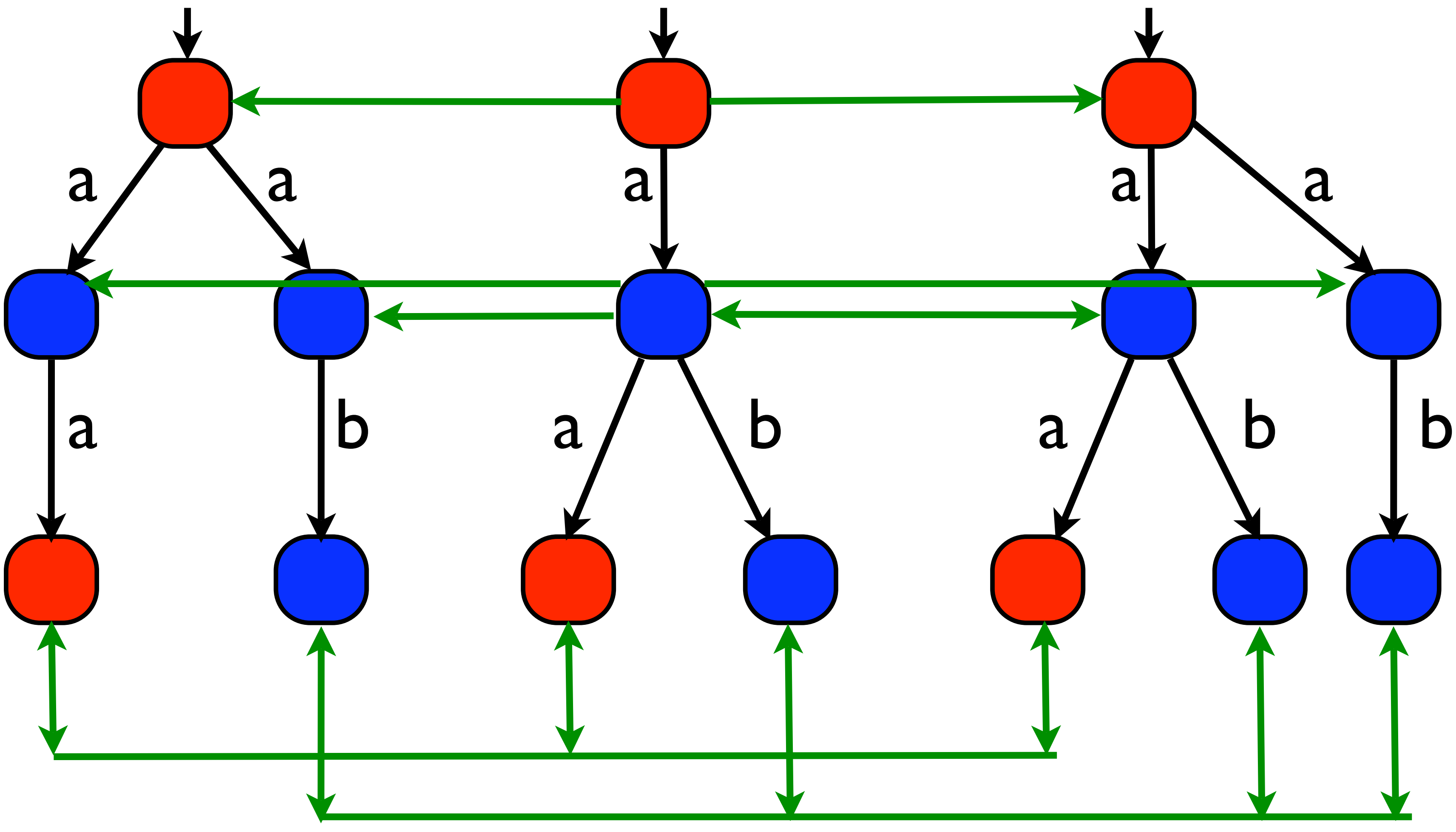


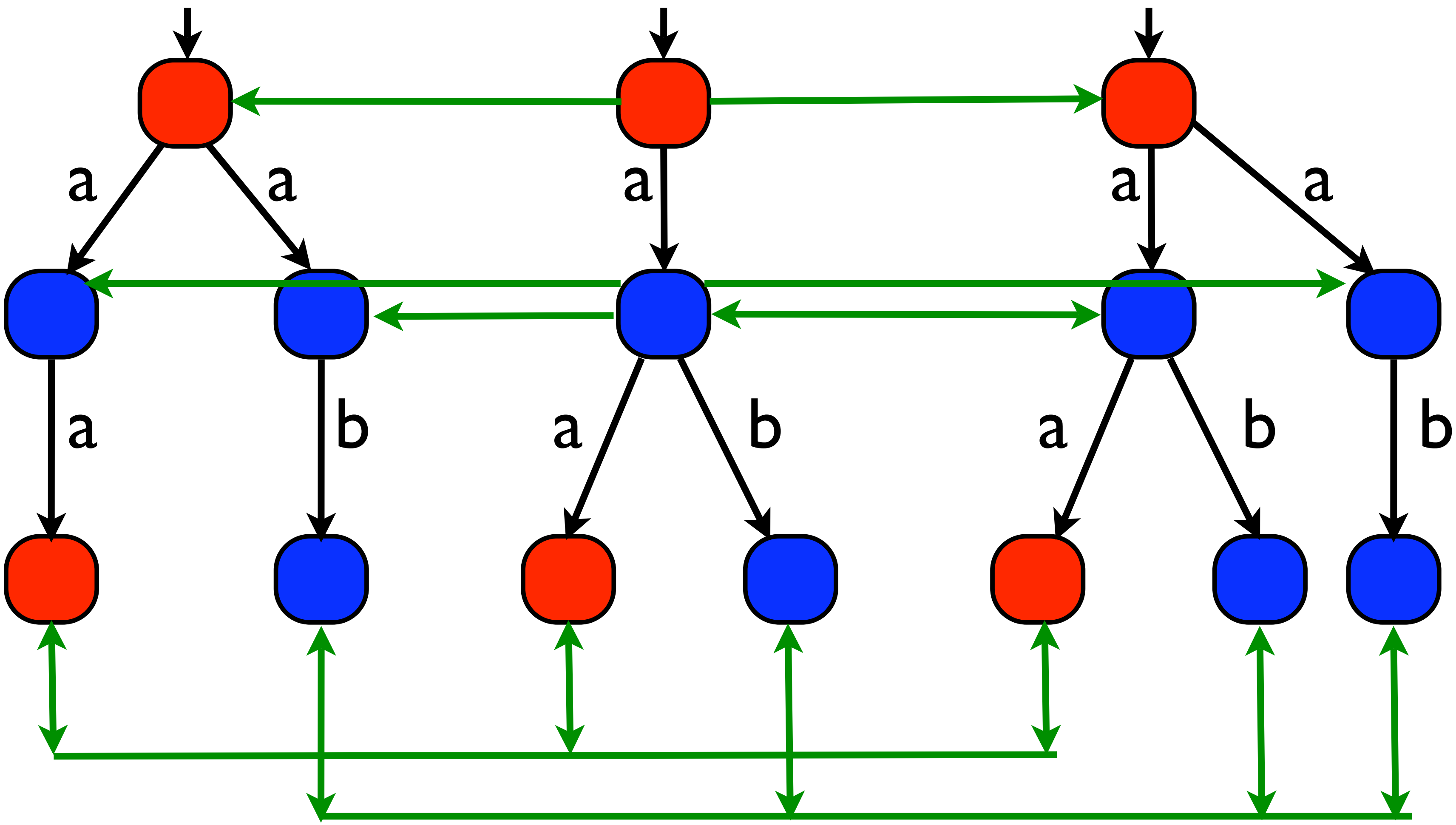
Who can simulate who ?

Simulation relations



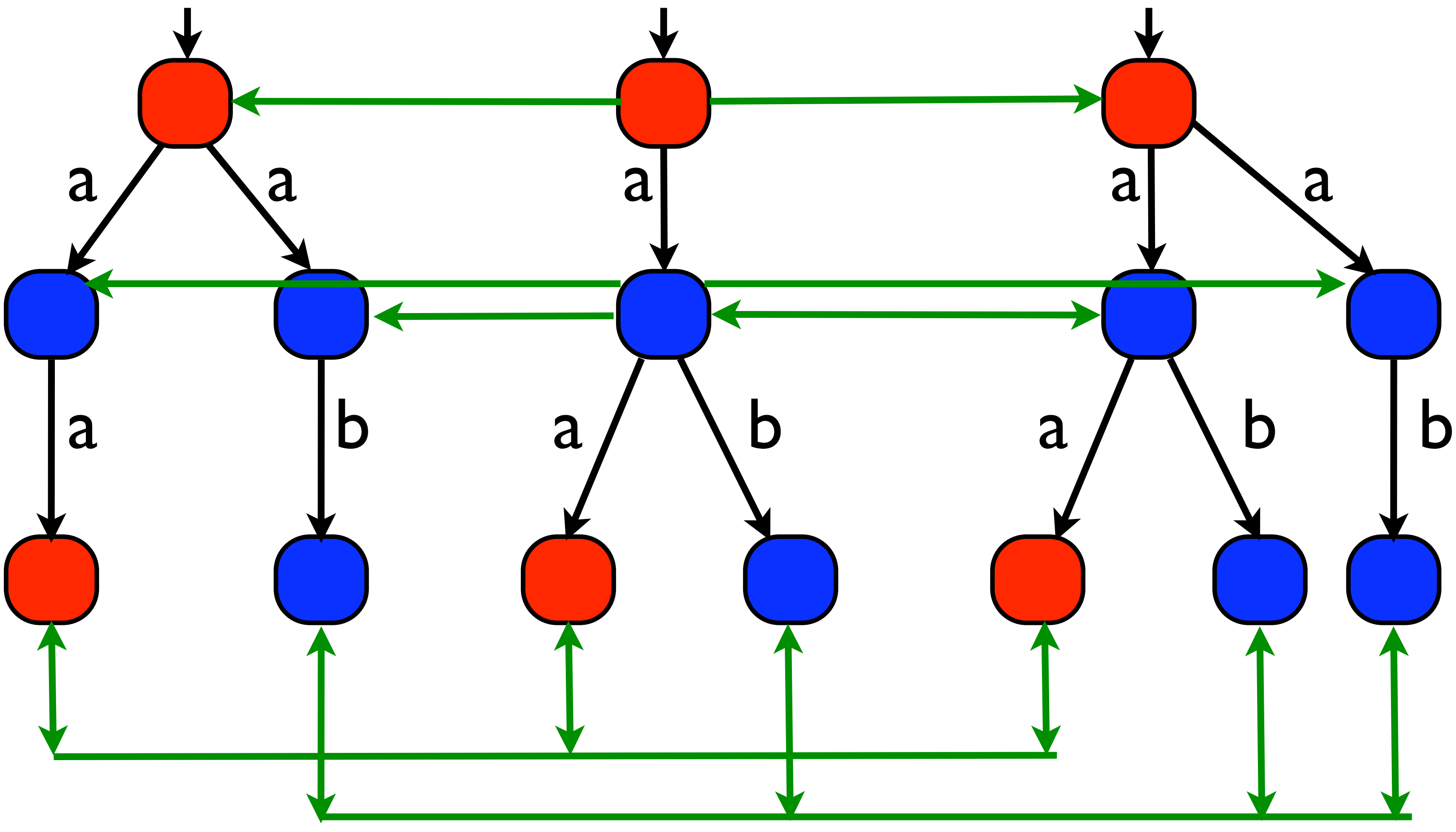
This is a simulation relation

Simulation relations



Is this the largest one ?

Simulation relations



Is this the largest one ? NO.

Simulation relations and bisimulations

Simulation relations and bisimulations

- Given a LTS $(S,S_0,\Sigma,T,C,\lambda)$, there exists a unique **largest** simulation relation $\mathbf{R}\subseteq S\times S$;

Simulation relations and bisimulations

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  for all s_1,s_2 such that $\mathbf{R}(s_1,s_2)$ we have also $\mathbf{R}(s_2,s_1)$;

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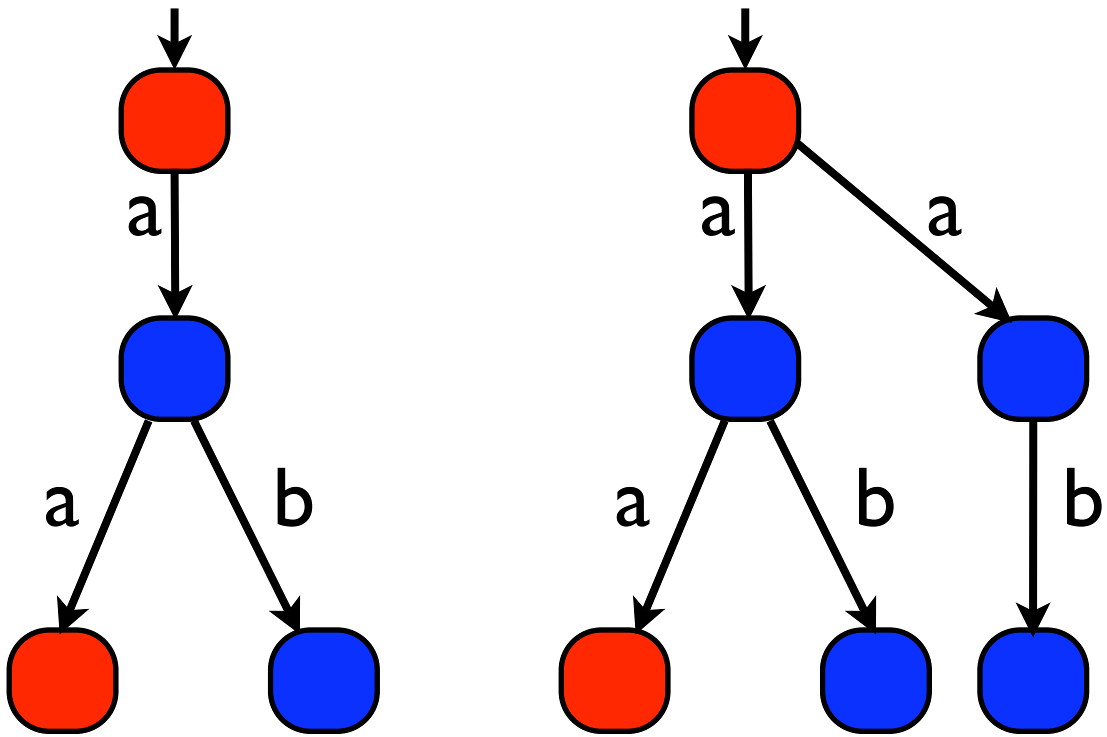
- A simulation relation **R** which is **symmetric** is called a **bisimulation**.

- Given a bisimulation relation **R** and two states s_1, s_2 such that $\mathbf{R}(s_1, s_2)$ (note that we have also $\mathbf{R}(s_2, s_1)$ by definition), we say that s_1 and s_2 are **bisimilar**, this is noted $s_1 \approx_{\mathbf{R}} s_2$ (or $s_1 \approx s_2$ if **R** is clear from the context).
  The relation $\approx_{\mathbf{R}}$ is an **equivalence relation**.

Bisimulation



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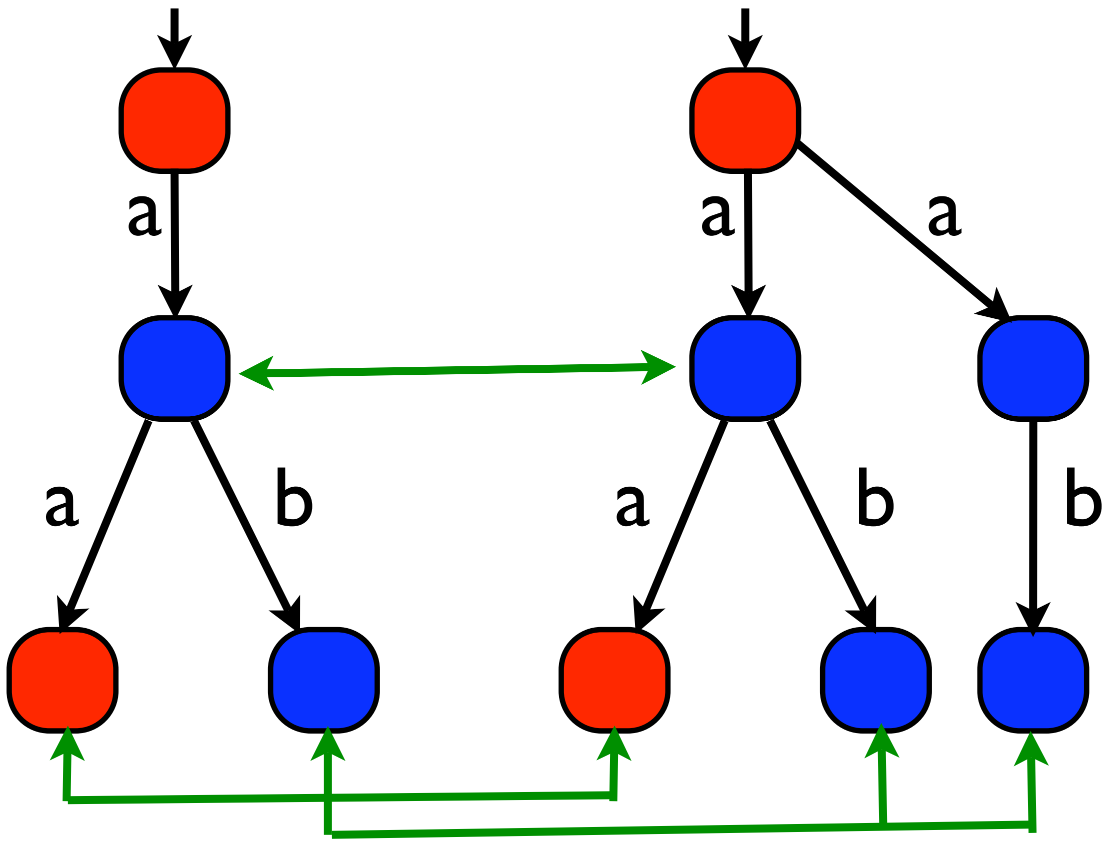


This is a bisimulation

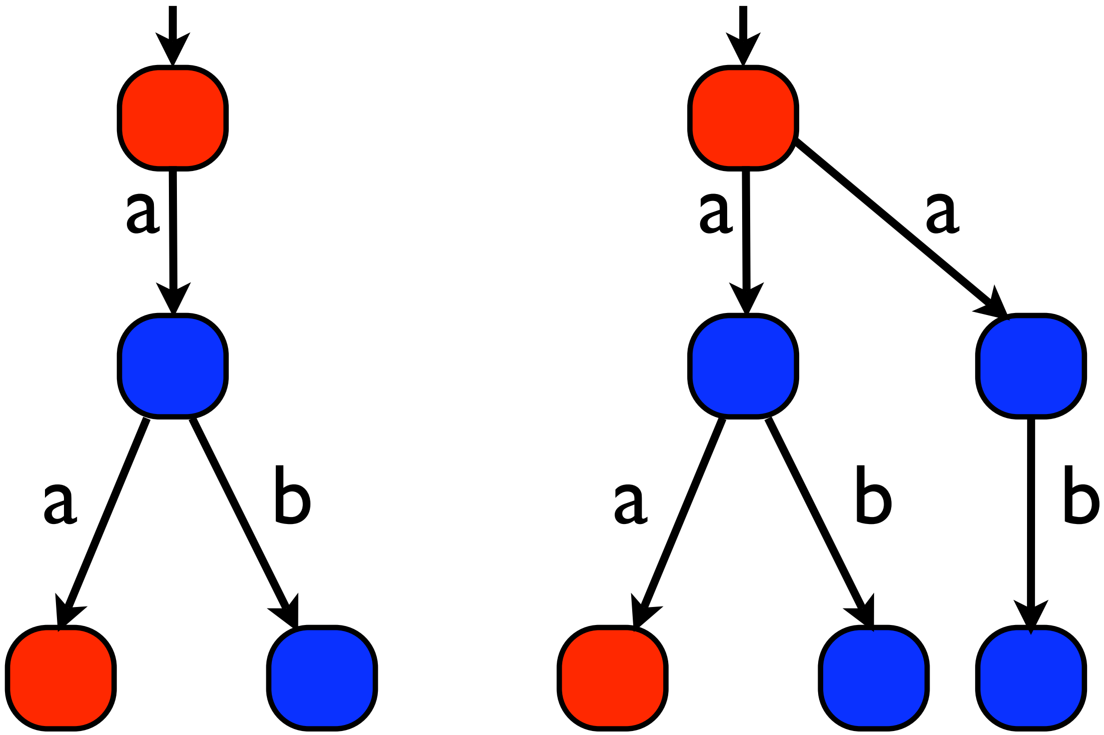
Quotient of a LTS using bisimulation

- Let $L=(S,S_0,\Sigma,T,C,\lambda)$ be a LTS, $\mathbf{R}\subseteq S\times S$ be a **bisimulation relation** over the state space of L, and let $\approx_{\mathbf{R}}$ be the associated equivalence relation.

- The **quotient by $\approx_{\mathbf{R}}$** of L is the LTS $L_\approx=(S_\approx,S_{0\approx},\Sigma,T_\approx,C,\lambda_\approx)$:
  - ➢ $S_\approx$ are the equivalence classes for $\approx_{\mathbf{R}}$;
  - ➢ $S_{0\approx}$ are the equivalence classes $\mathbf{s}$ for $\approx_{\mathbf{R}}$ such that for all $s\in\mathbf{s}$, $s\in S_0$;
  - ➢ $T_\approx$ is such that $T_\approx(\mathbf{s}_1,\sigma,\mathbf{s}_2)$ iff $\exists s_1\in\mathbf{s}_1, s_2\in\mathbf{s}_2: T(s_1,\sigma,s_2)$;
  - ➢ $\lambda_\approx$ is such that $\lambda_\approx(\mathbf{s})=\lambda(s)$ for any $s\in\mathbf{s}$.

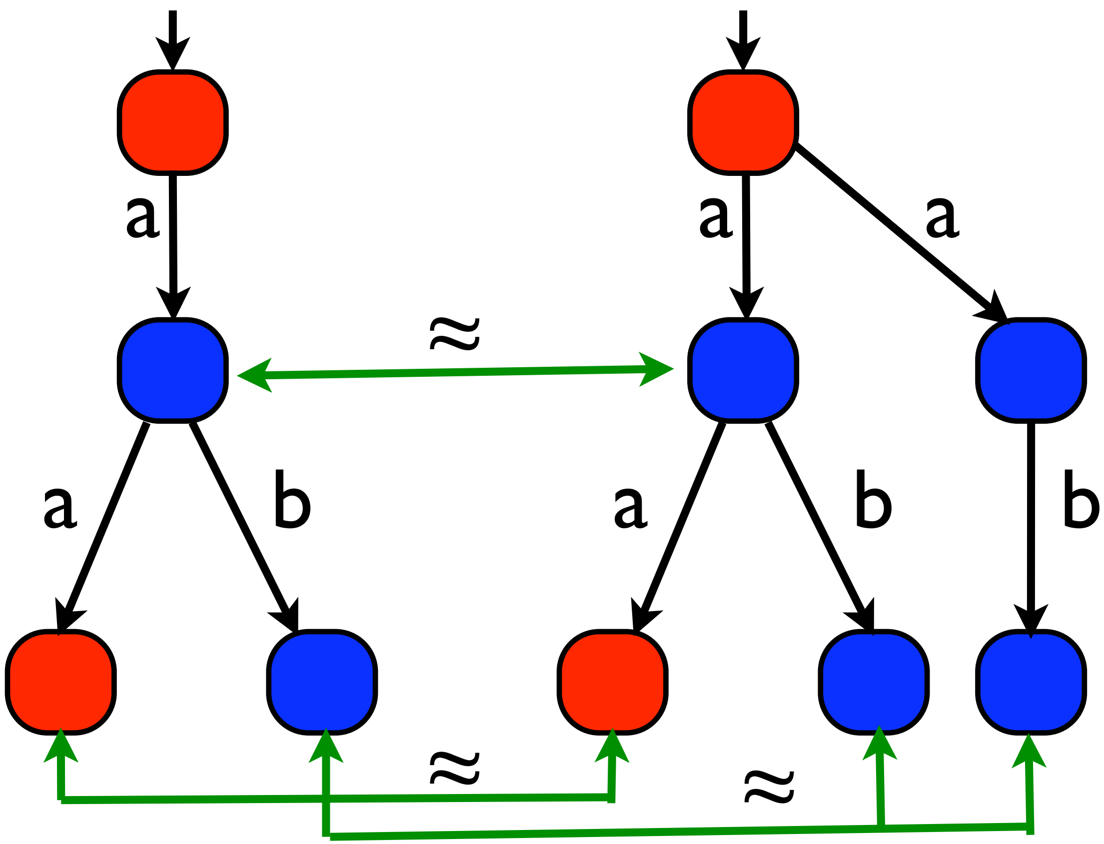
- **Theorem**:
  Let L be a LTS and **R** a bisimulation over the state space of L, let $L_\approx$ be the quotient of L by $\approx_{\mathbf{R}}$, then **Traces**$(L)=$**Traces**$(L_\approx)$.

Quotient of a LTS using bisimulation



The LTS

Quotient of a LTS using bisimulation



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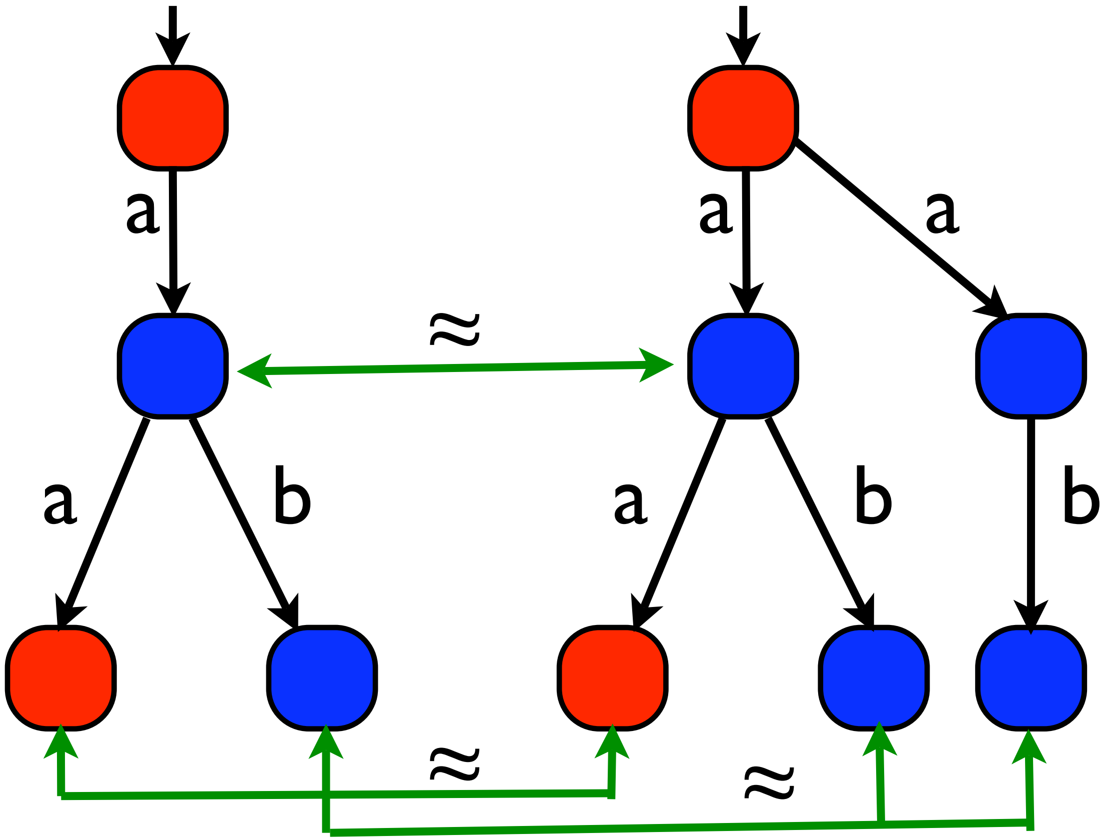

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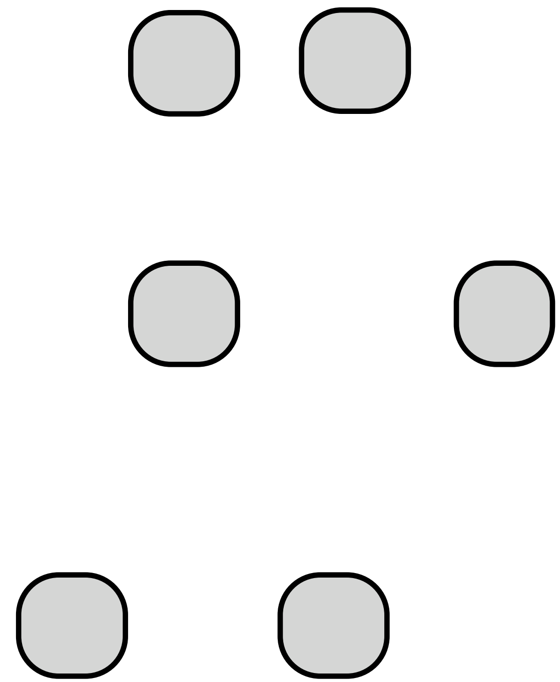

The quotient by ≈

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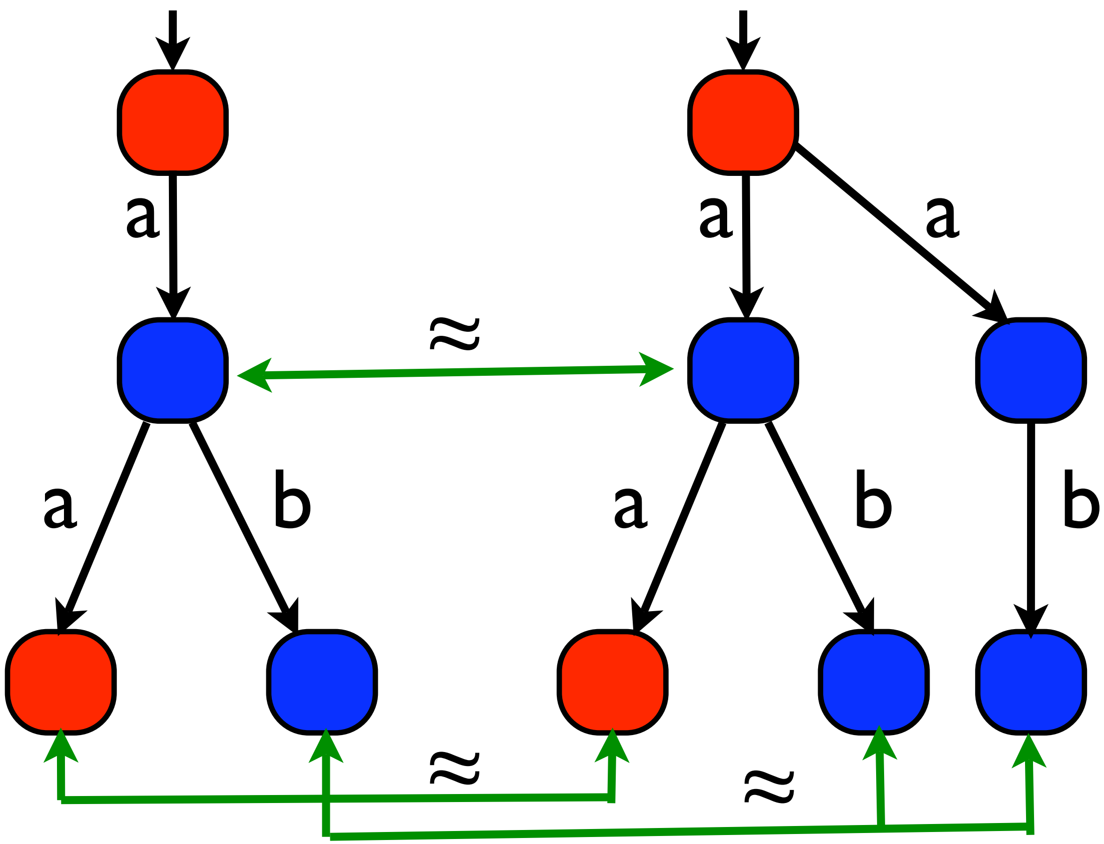

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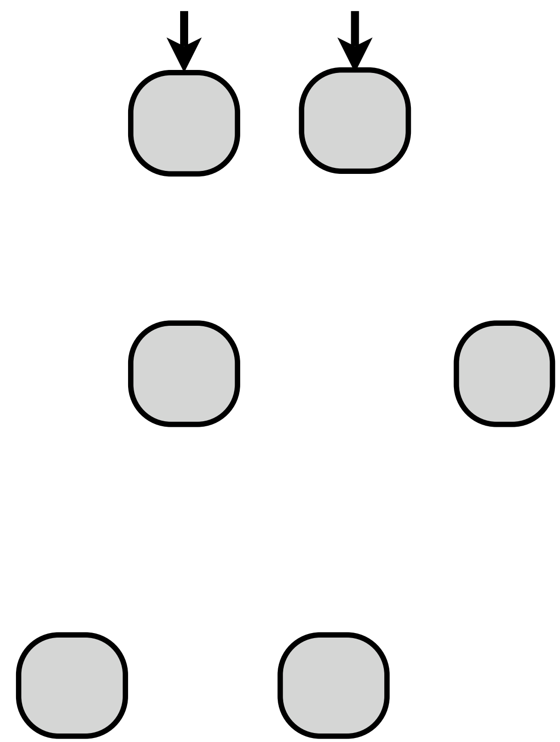

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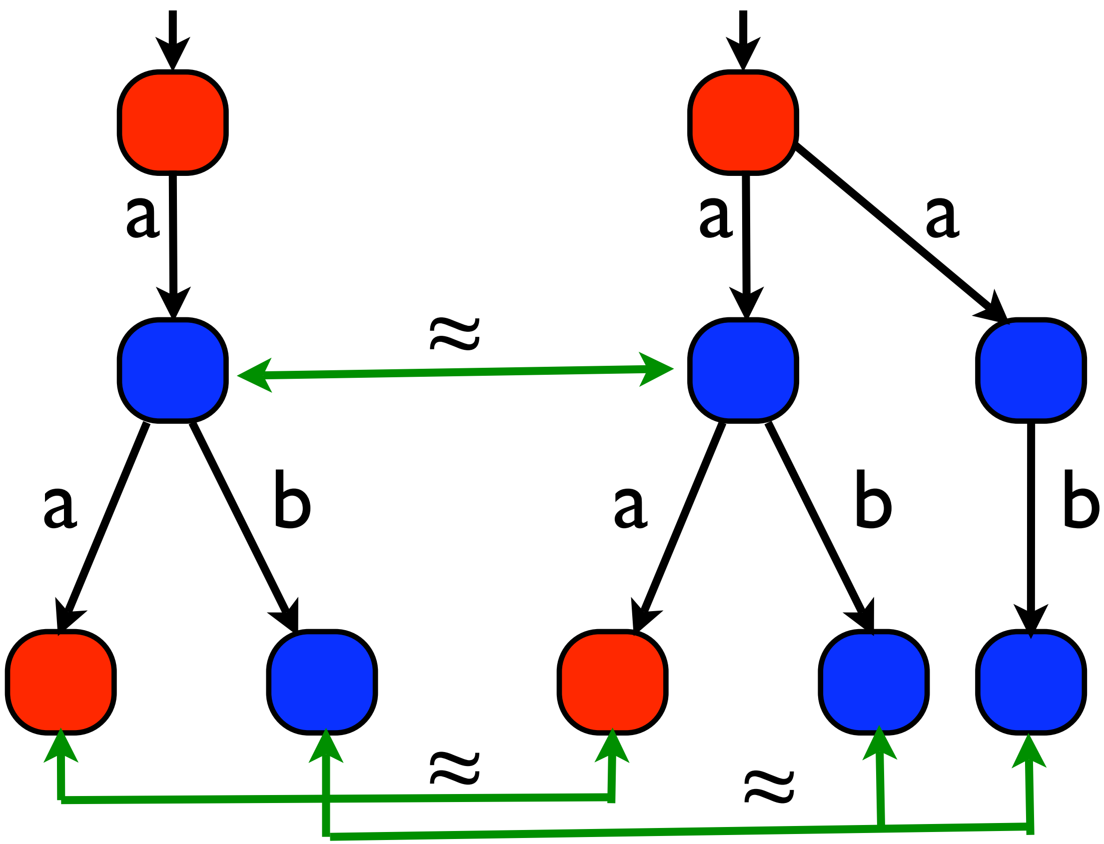

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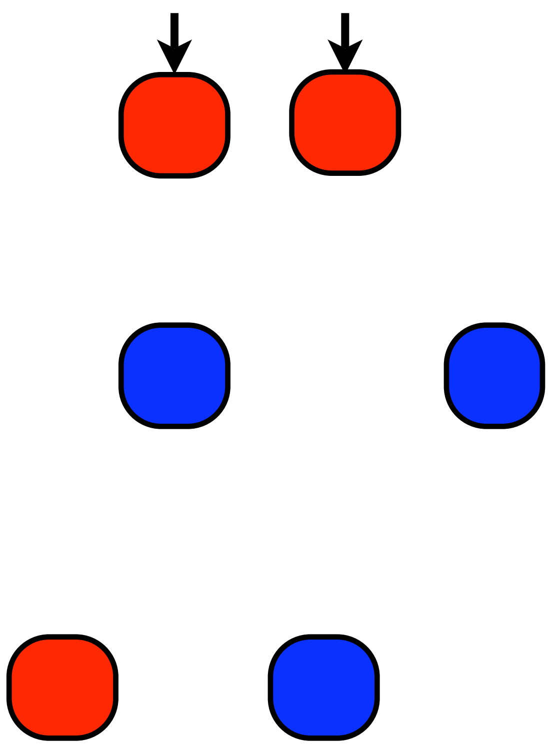

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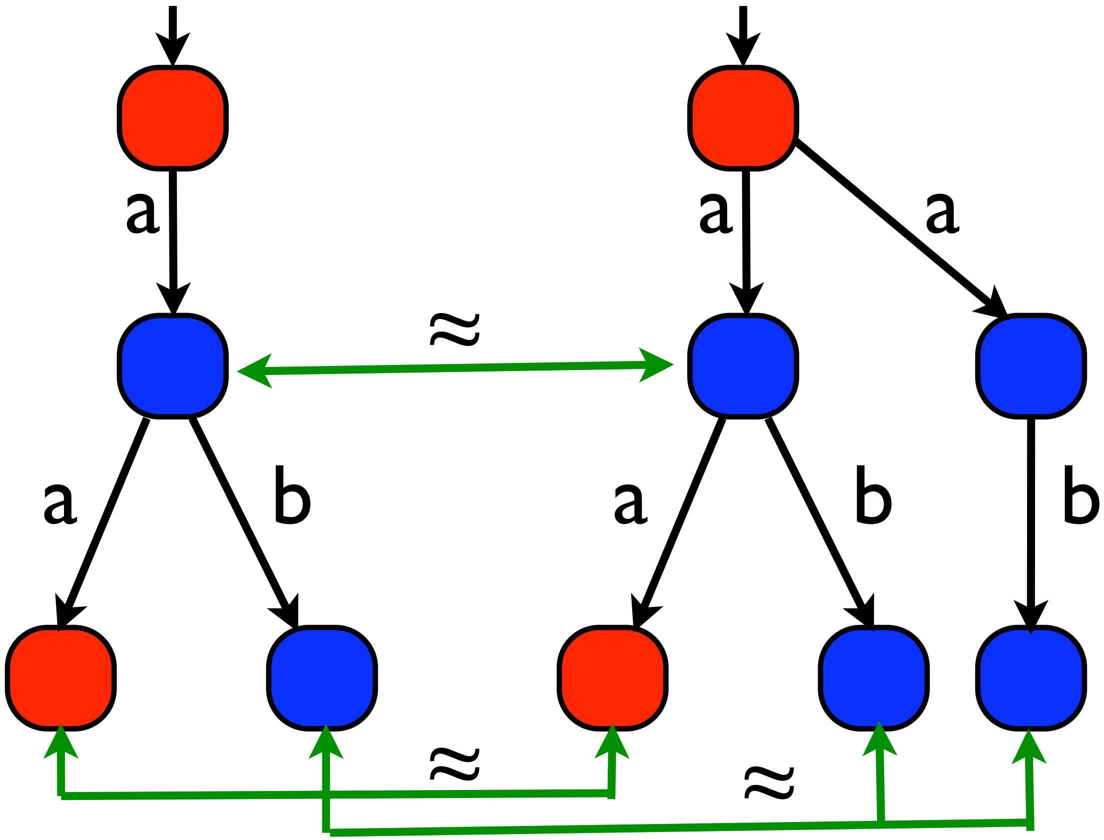

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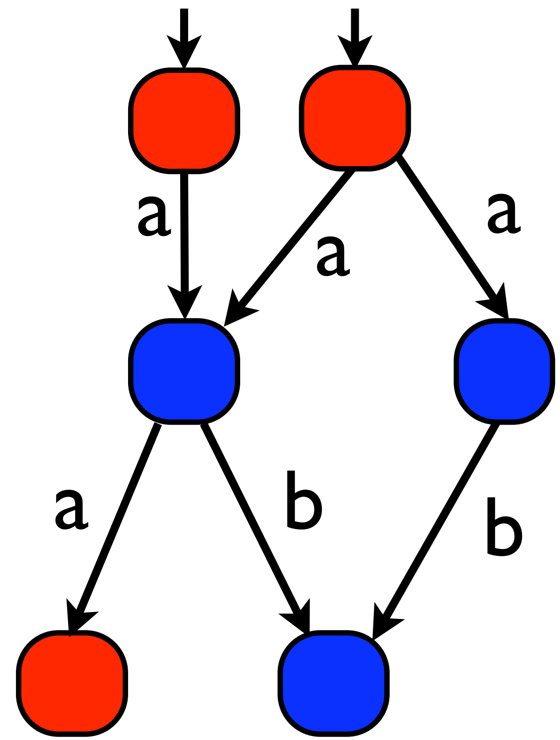
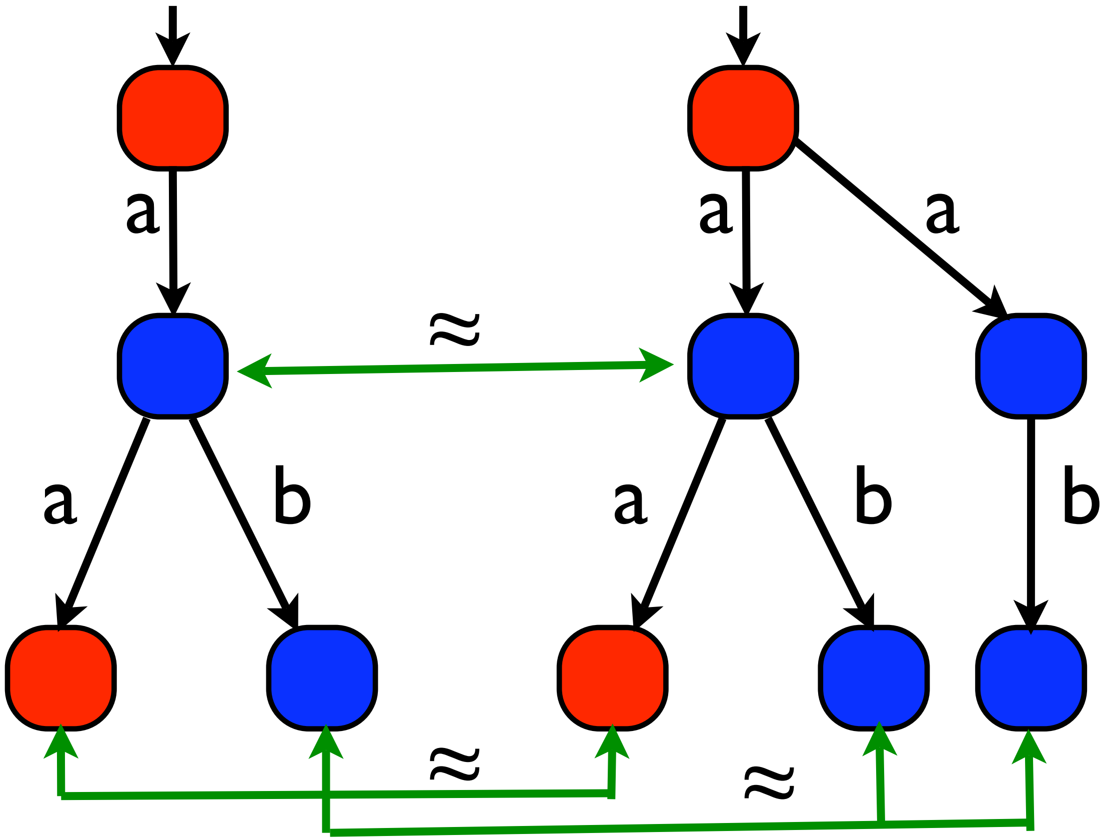


The quotient by ≈

Bisimulation is not complete for trace equivalence



The LTS

Clearly, the two initial states
are trace equivalent
but they are not bisimilar.

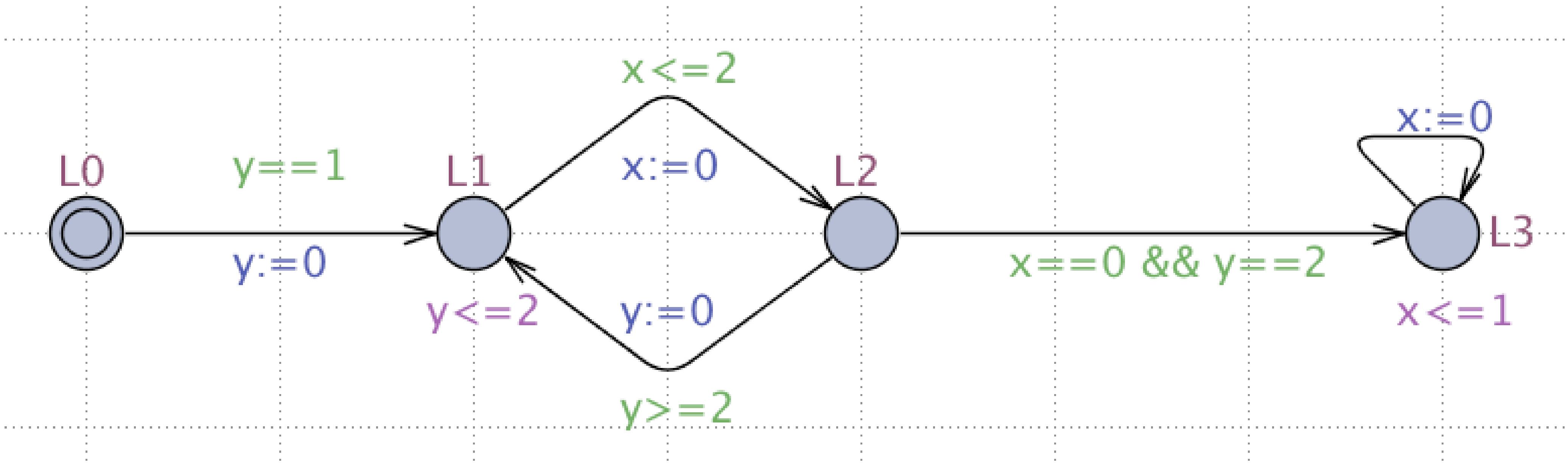
Plan of the talk

- Labelled transition systems
- Properties of labeled transition systems:
  Reachability - Safety - Büchi properties
- Pre-Post operators
- Partial orders - Fixed points
- Symbolic model-checking
- Application to TA: region equivalence, region automata, zones

Algorithmic verification of timed automata

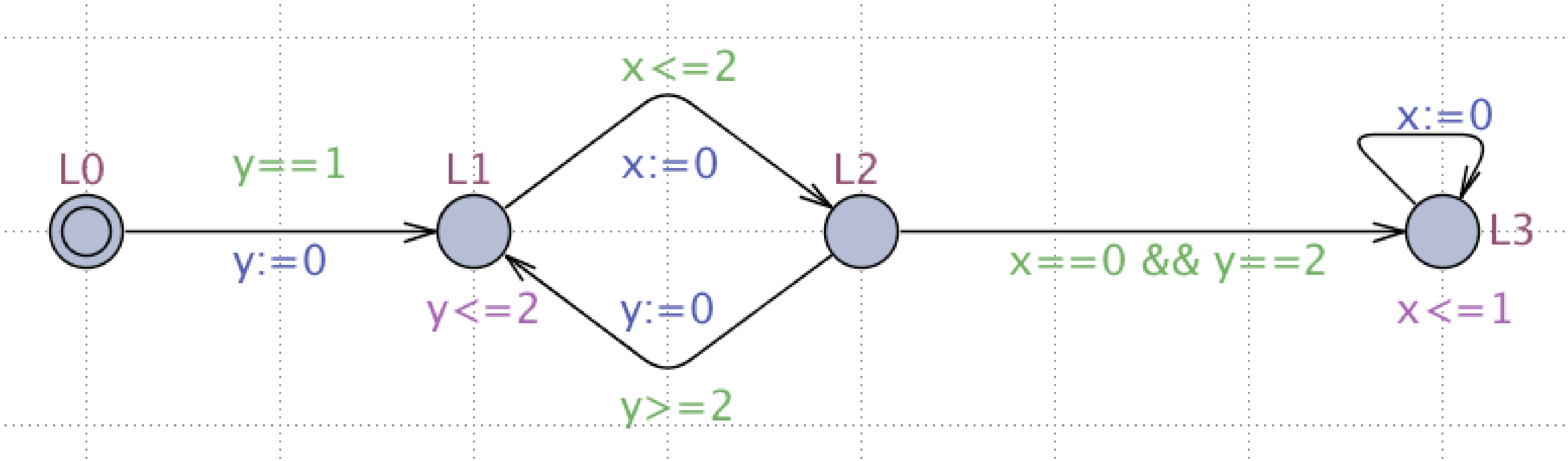
- We will show now how to apply the concepts that we have introduced so far to obtain algorithms to verify properties of timed automata;
- We will show how to use the **pre-post operators** to build **fixed points algorithms**;
- We will show that those algorithms are terminating by showing that they operate over **finite state time-abstract bisimulation quotients**.

A timed automaton



Question: Can L3 be reached ?

A timed automaton



Question: Can L3 be reached ?

This question can be reduced to a reachability verification problem over the labeled transition system of the TA.

Labeled transition system of a TA

- The $LTS=(S,S_0,\Sigma,T,C,\lambda)$ of a TA $A=(Q,Q_0,\Sigma,P,Cl,E,L,F,Inv)$, is as follows:

  - S is the set of pairs (q,v) where $q \in Q$ is a location of A and $v : Cl \to \mathbb{R}_{\geq 0}$ such that $v \vDash Inv(q)$;

  - $S_0=\{(q_0,<0,0,\ldots 0,>) \mid q_0 \in Q_0\}$;

  - $T \subseteq S \times (\Sigma \cup \mathbb{R}_{\geq 0}) \times S$ defined by two types of transitions:

    Discrete transitions:
  $(q_1,v_1) \to_a (q_2,v_2) \in T$ iff there exists $(q_1,a,\Phi,\Delta,q_2) \in E$, $v_1 \vDash \Phi$, and $v_2:=v_1[\Delta:=0]$.
    Continuous transitions:
  $(q_1,v_1) \to_\delta (q_2,v_2) \in T$ iff $q_1=q_2$, $\delta \in \mathbb{R}_{\geq 0}$, $v_2=v_1+\delta$, and $\forall \delta', 0 \leq \delta' \leq \delta$, $v_1+\delta \vDash Inv(q_1)$.

  - $C=2^P$, $\lambda((q,v))=L(q)$, for any $(q,v) \in Q$.

- Clearly, this transition system has a (continuous) infinite number of states. How do we handle it ?

Time abstract-labeled transition system

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  - S is the set of pairs (q,v) where $q \in Q$ is a location of A and $v : Cl \to \mathbb{R}_{\geq 0}$ such that $v \models Inv(q)$;

  - $S_0=\{(q_0,<0,0,...0,>) \mid q_0 \in Q_0\}$;

  - $T \subseteq S \times (\Sigma \cup \{Delay\}) \times S$ defined by two types of transitions:

    Discrete transitions:
  $(q_1,v_1) \to_a (q_2,v_2) \in T$ iff there exists $(q_1,a,\Phi,\Delta,q_2) \in E$, $v_1 \models \Phi$, and $v_2:=v_1[\Delta:=0]$.
    Continuous transitions:
  $(q_1,v_1) \to_{Delay} (q_2,v_2) \in T$ iff $q_1=q_2$, $\exists \delta \in \mathbb{R}_{\geq 0}$, $v_2=v_1+\delta$, and $\forall \delta'$, $0 \leq \delta' \leq \delta$, $v_1+\delta \models Inv(q_1)$.

  - $C=2^P$, $\lambda((q,v))=L(q)$, for any $(q,v) \in Q$.

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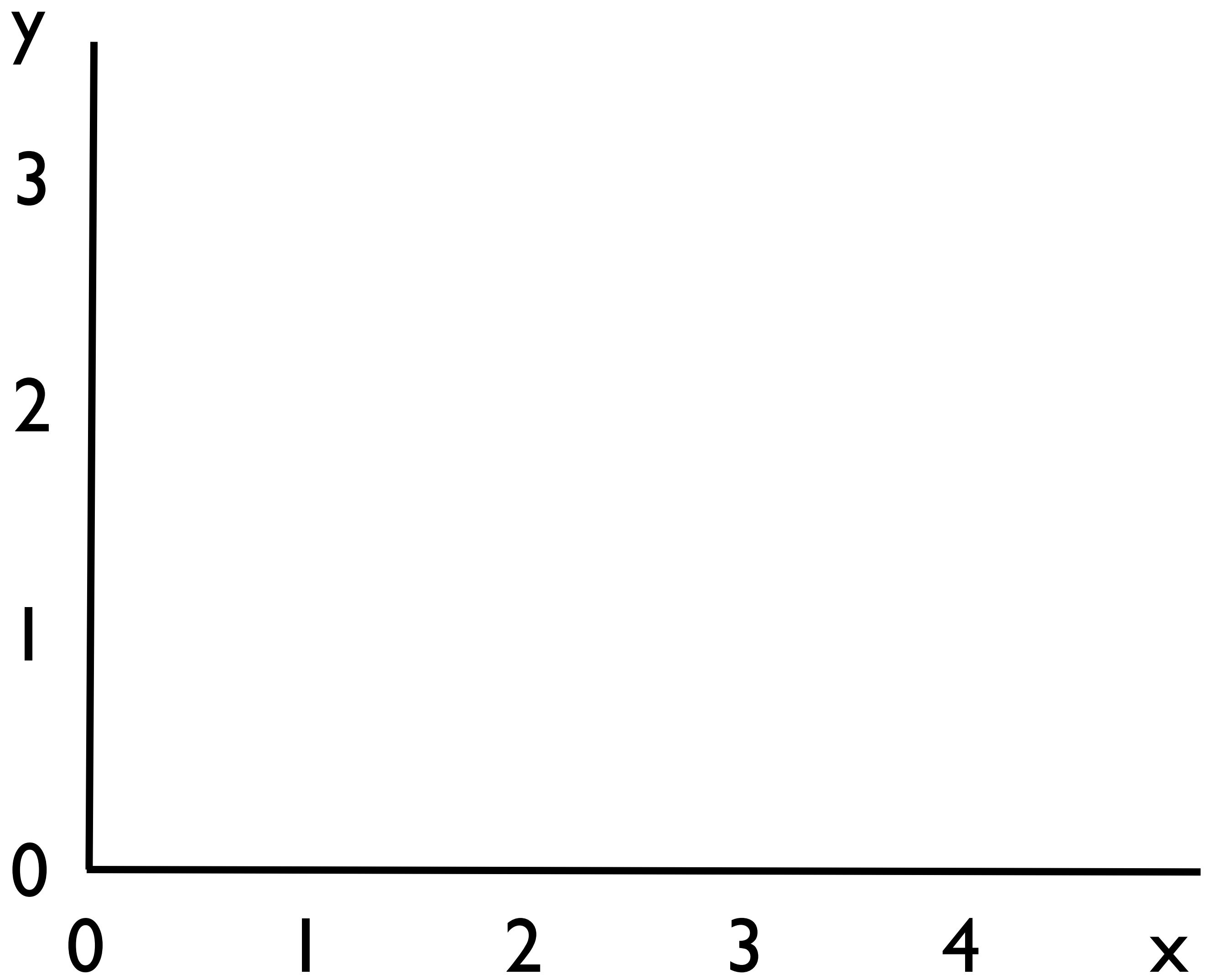
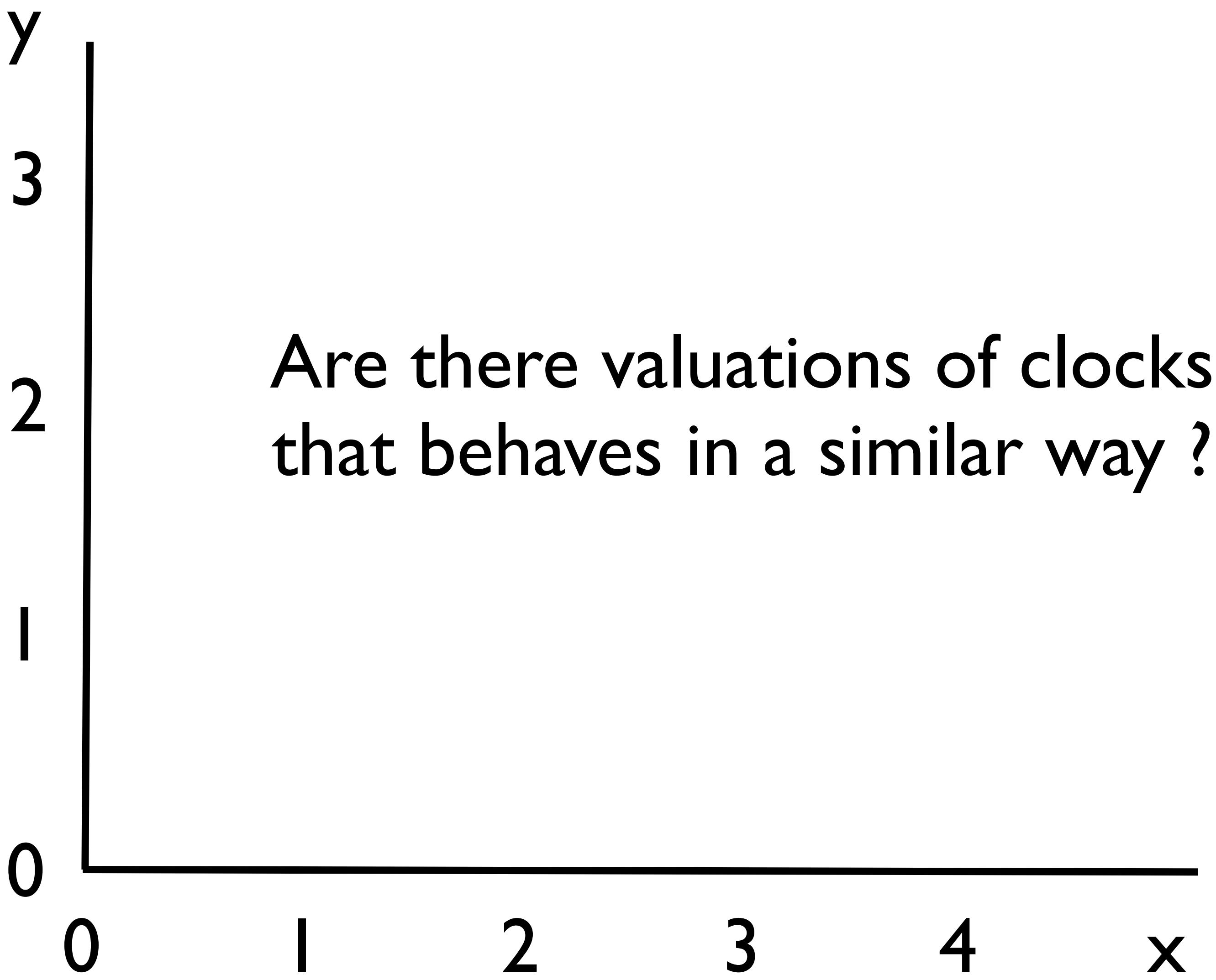
Continuous state space

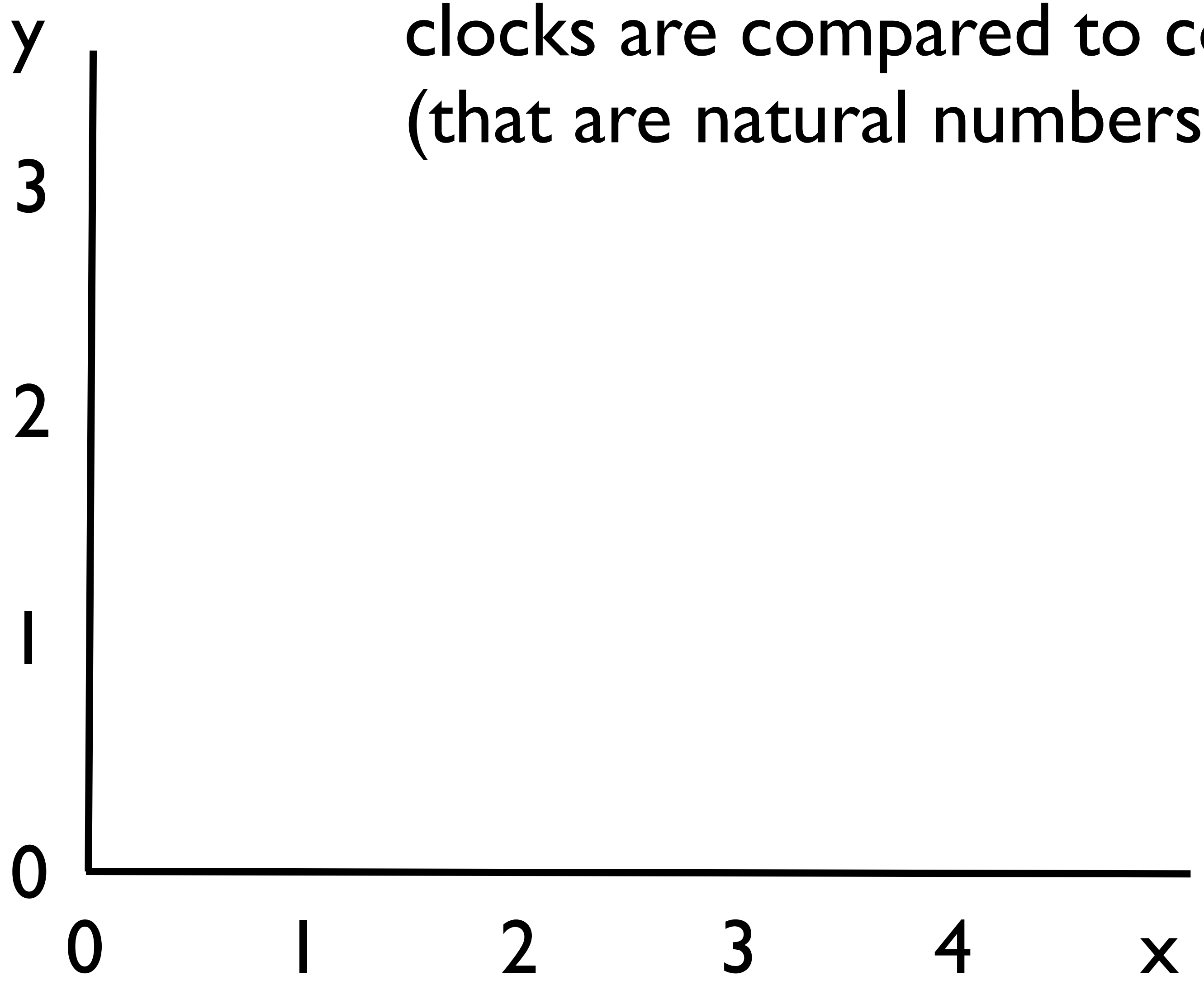


Continuous state space



Continuous state space

First, let us note that clocks are compared to constants (that are natural numbers).



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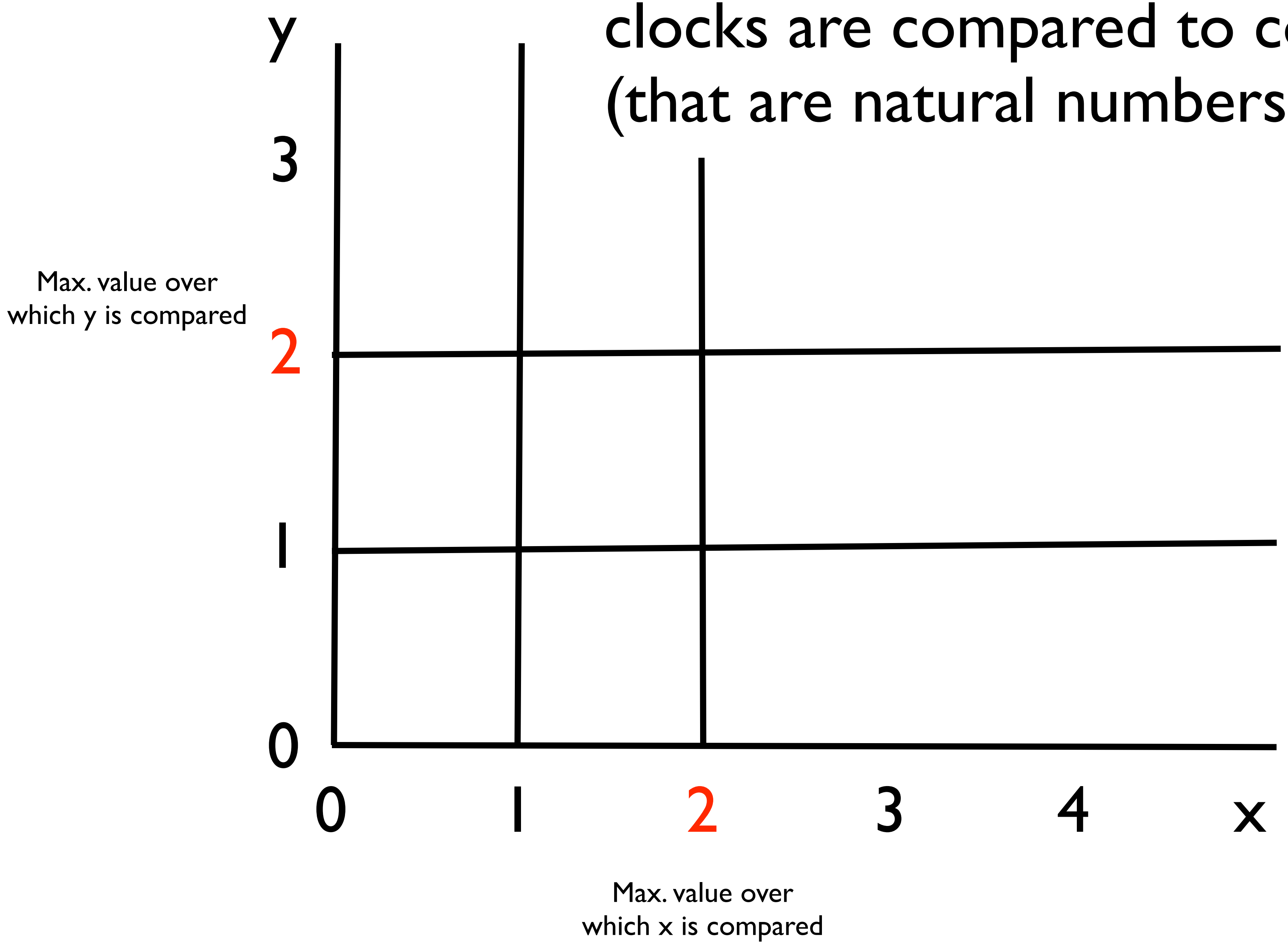
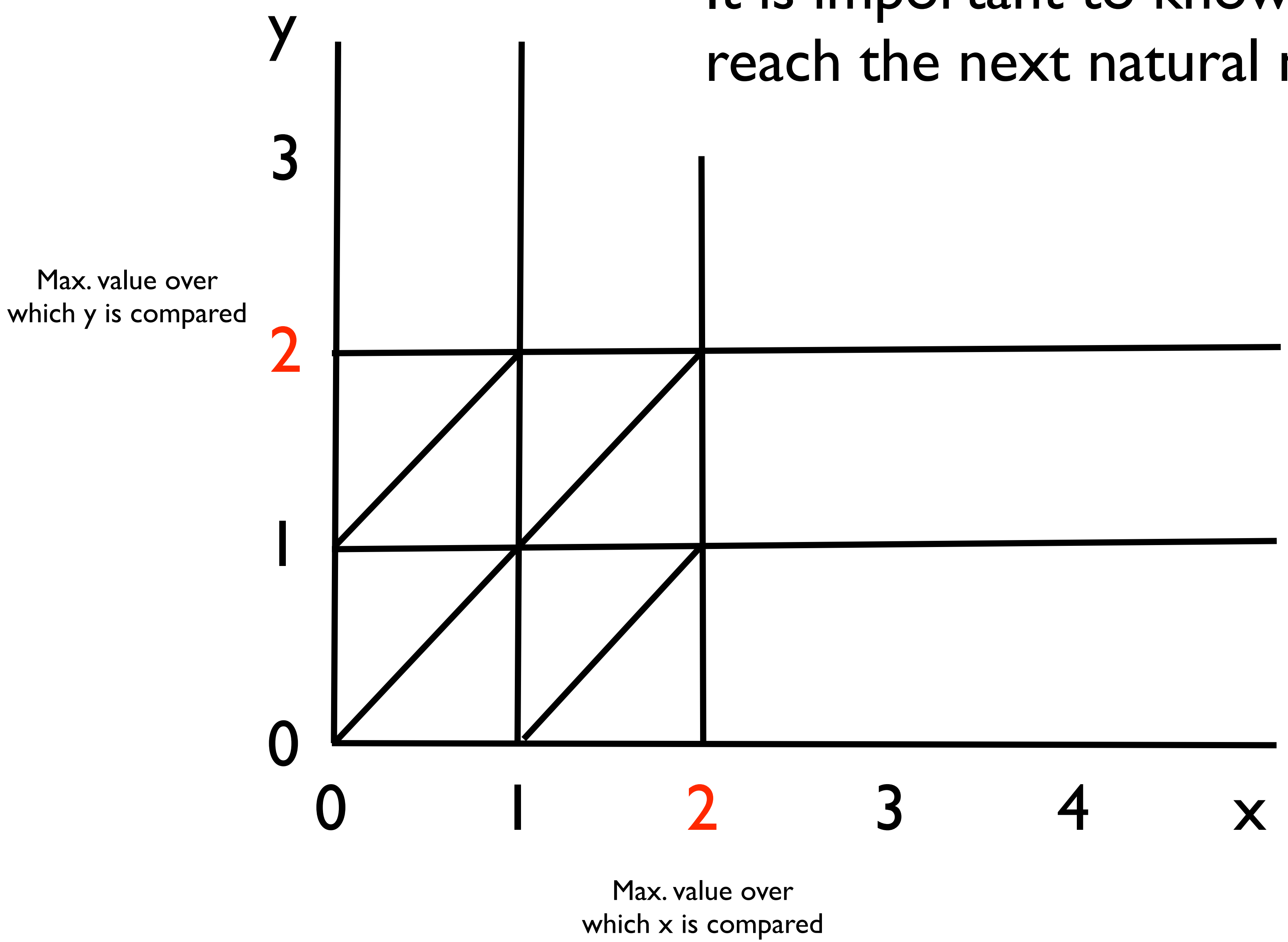


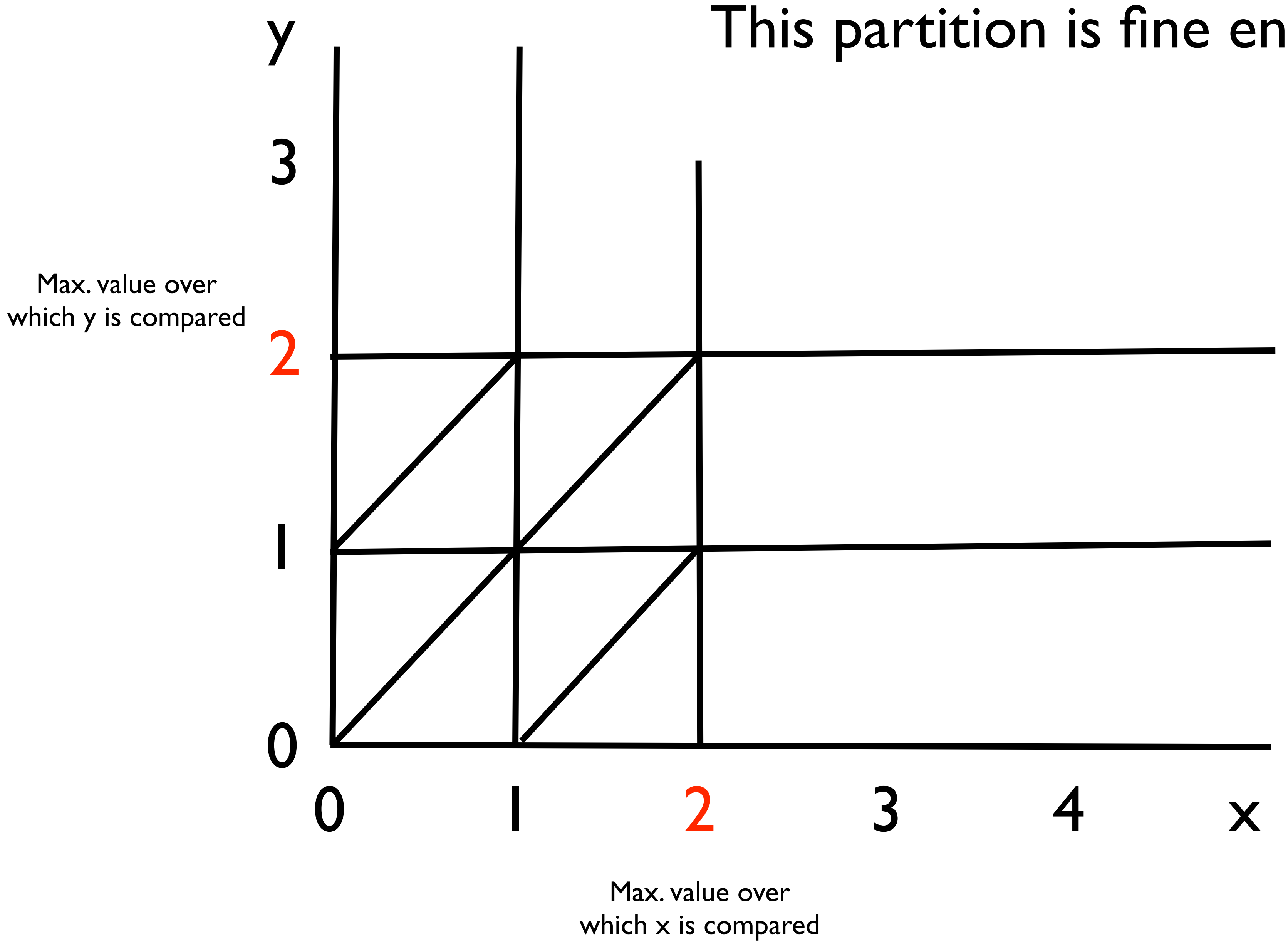
Continuous state space

It is important to know if x or y reach the next natural number first.



Continuous state space

This partition is fine enough



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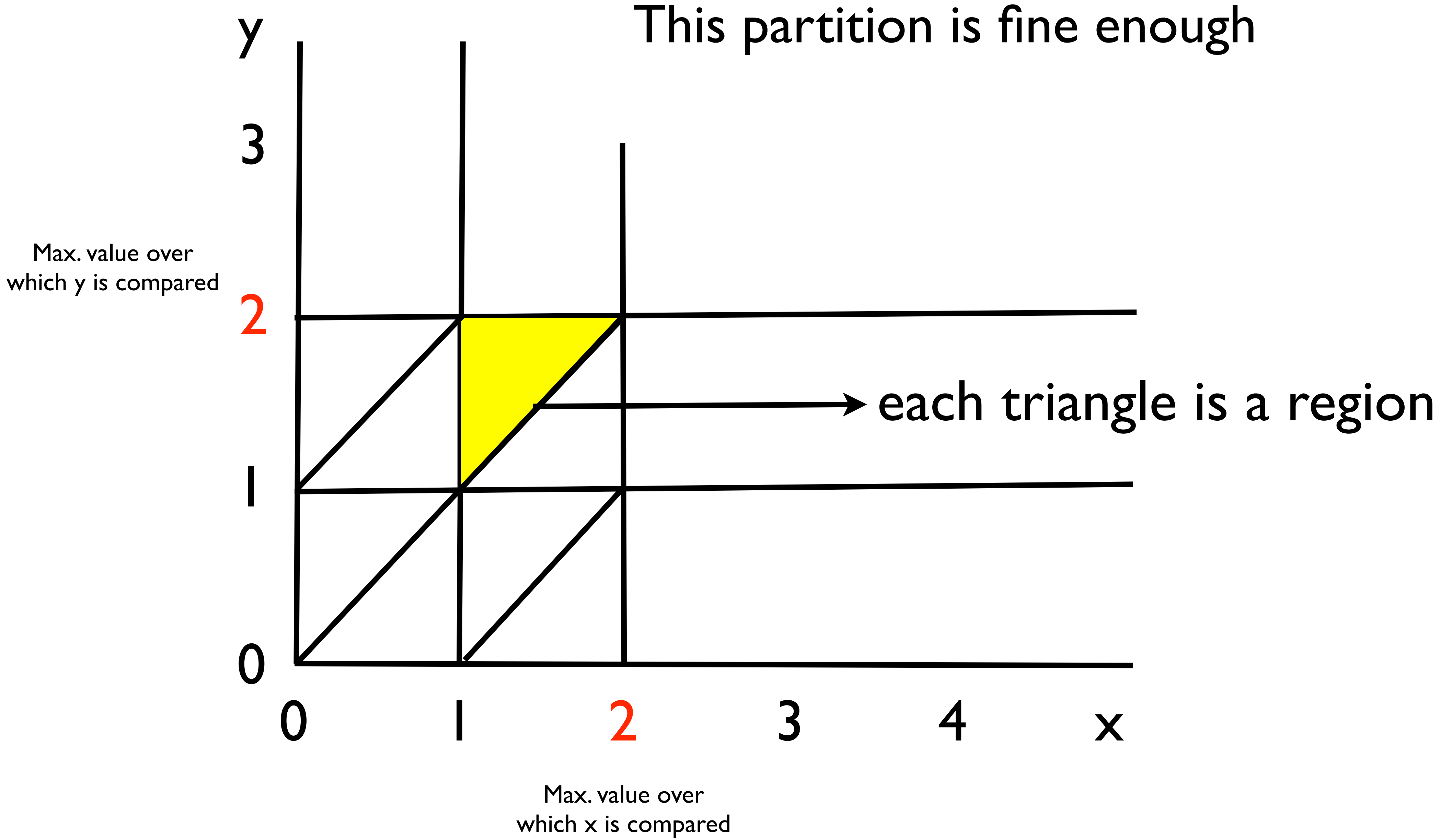
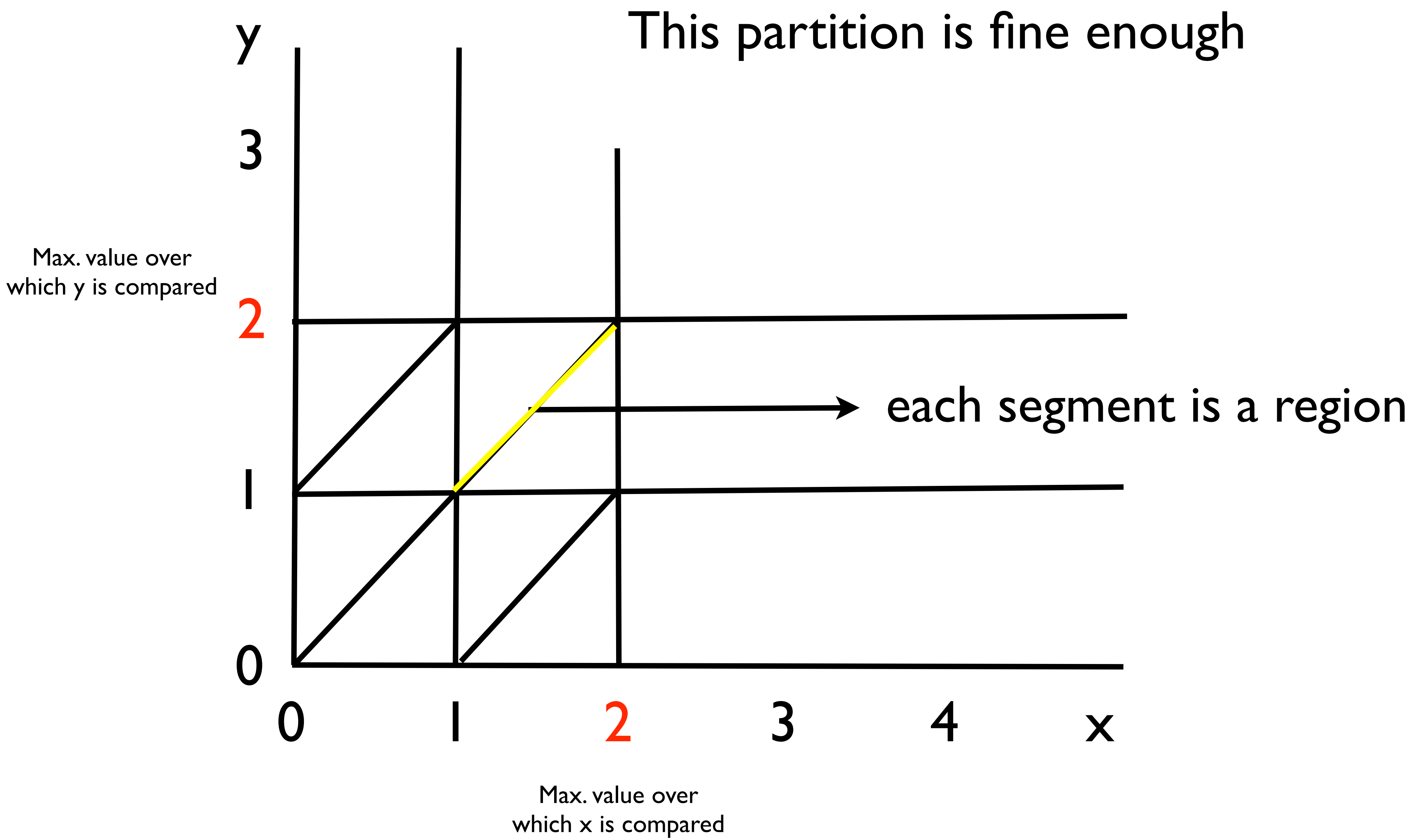


Continuous state space



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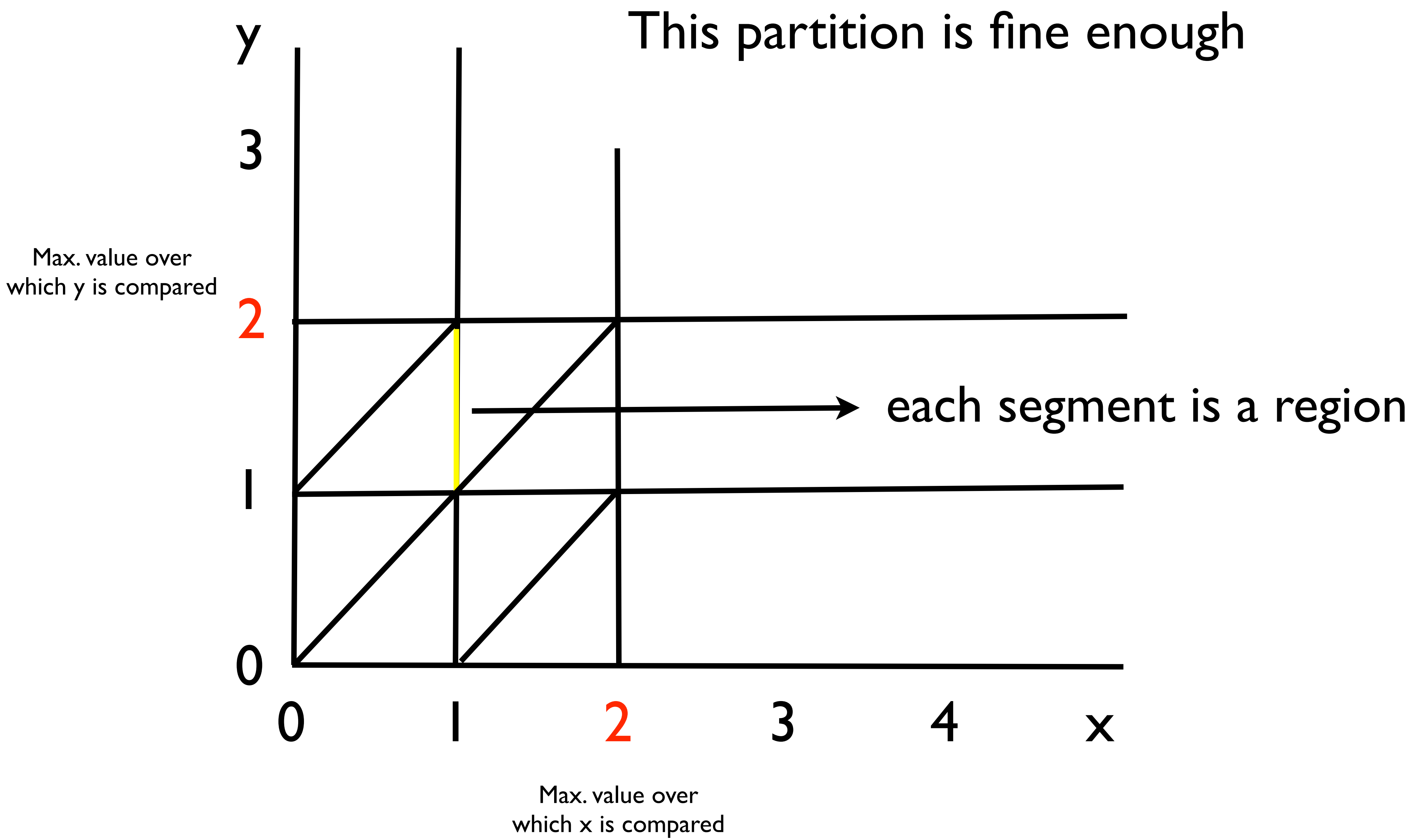

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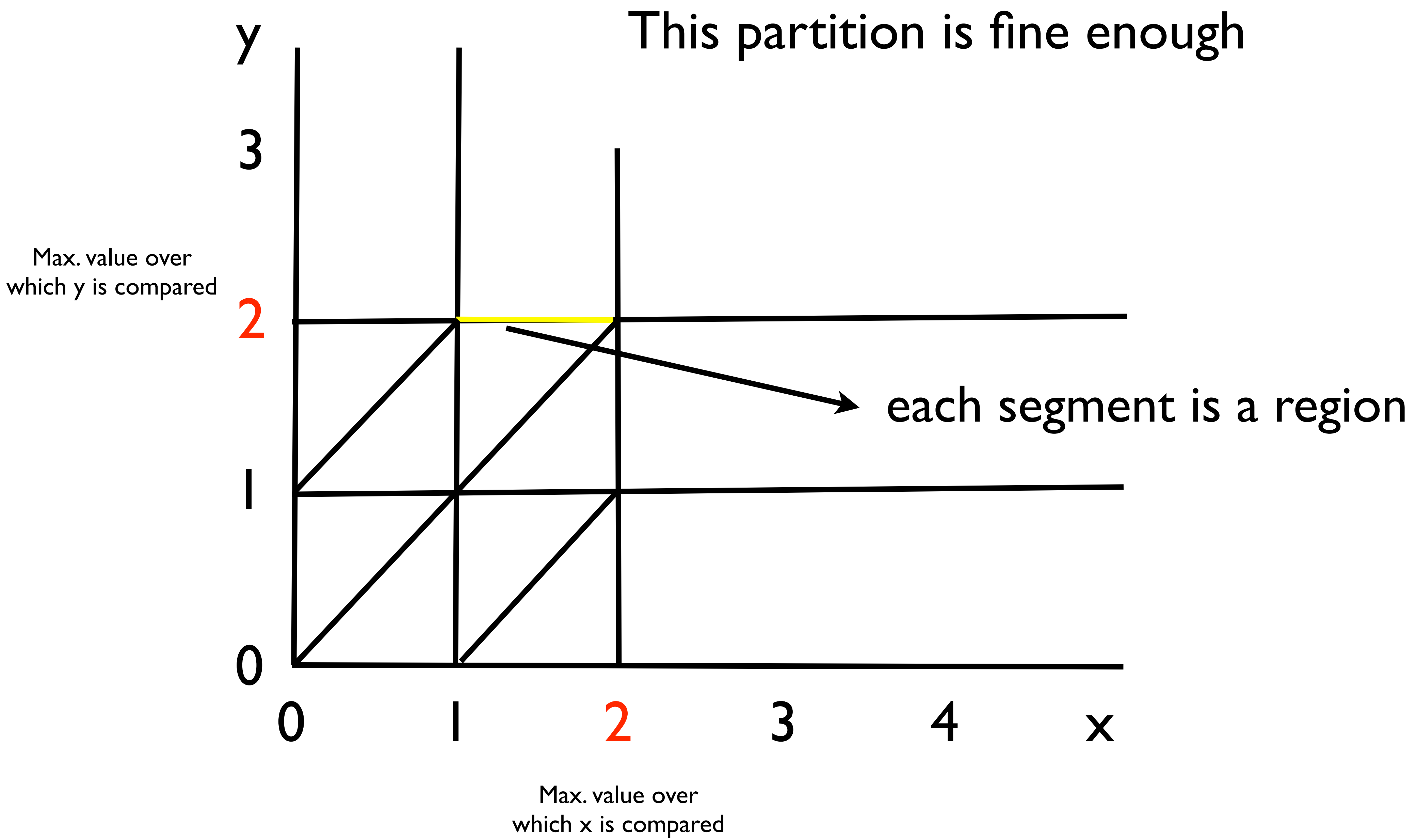
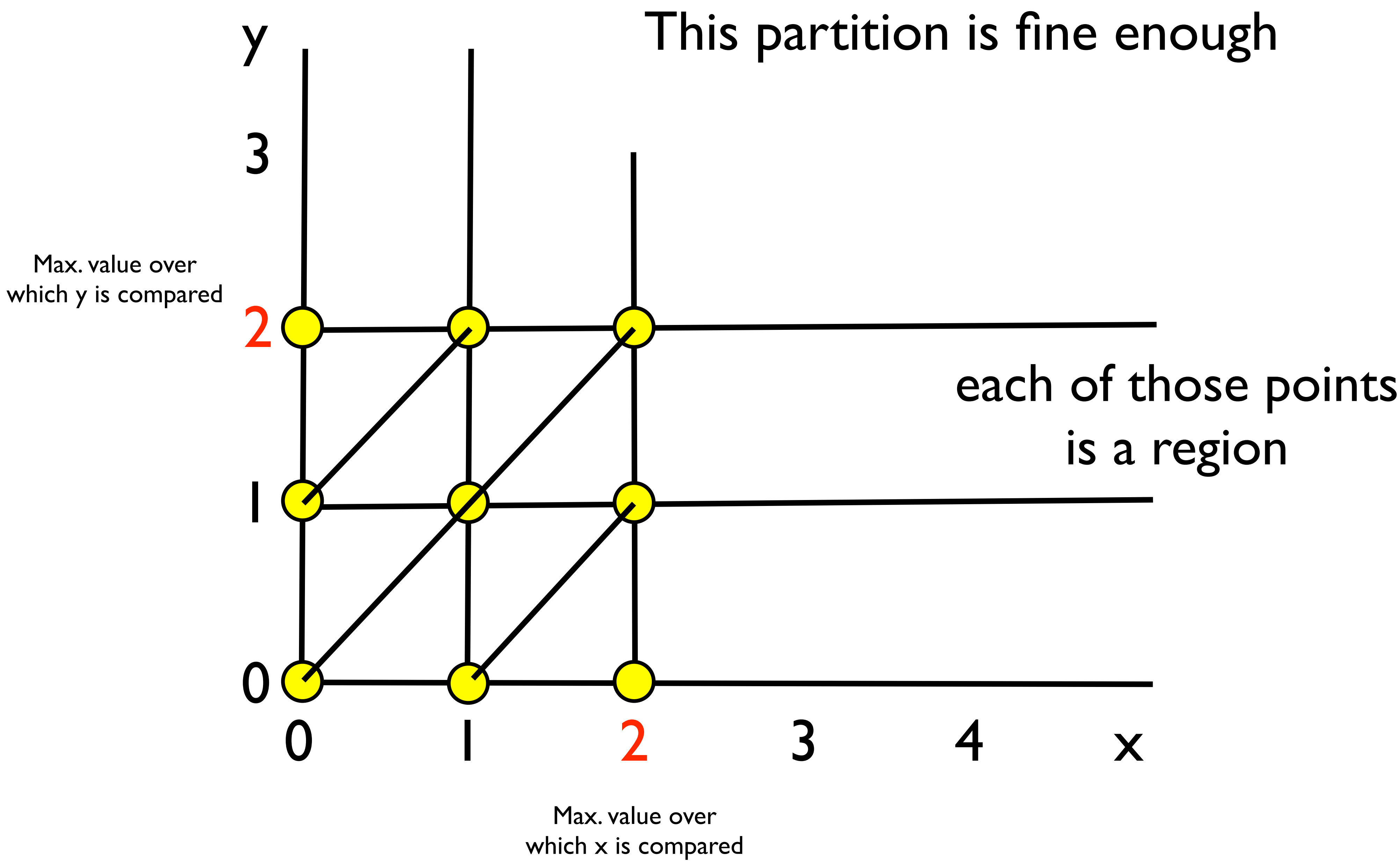


Continuous state space



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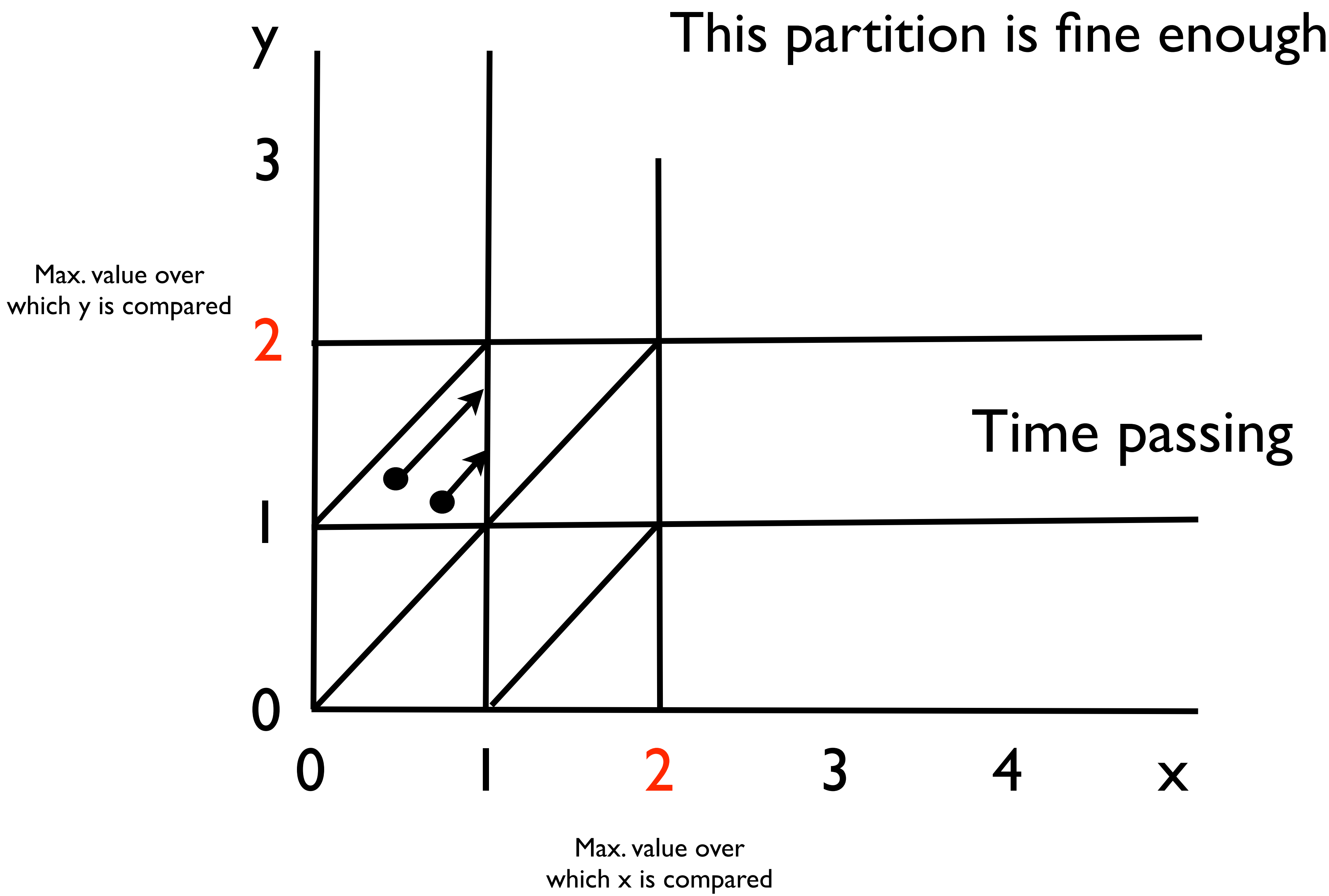

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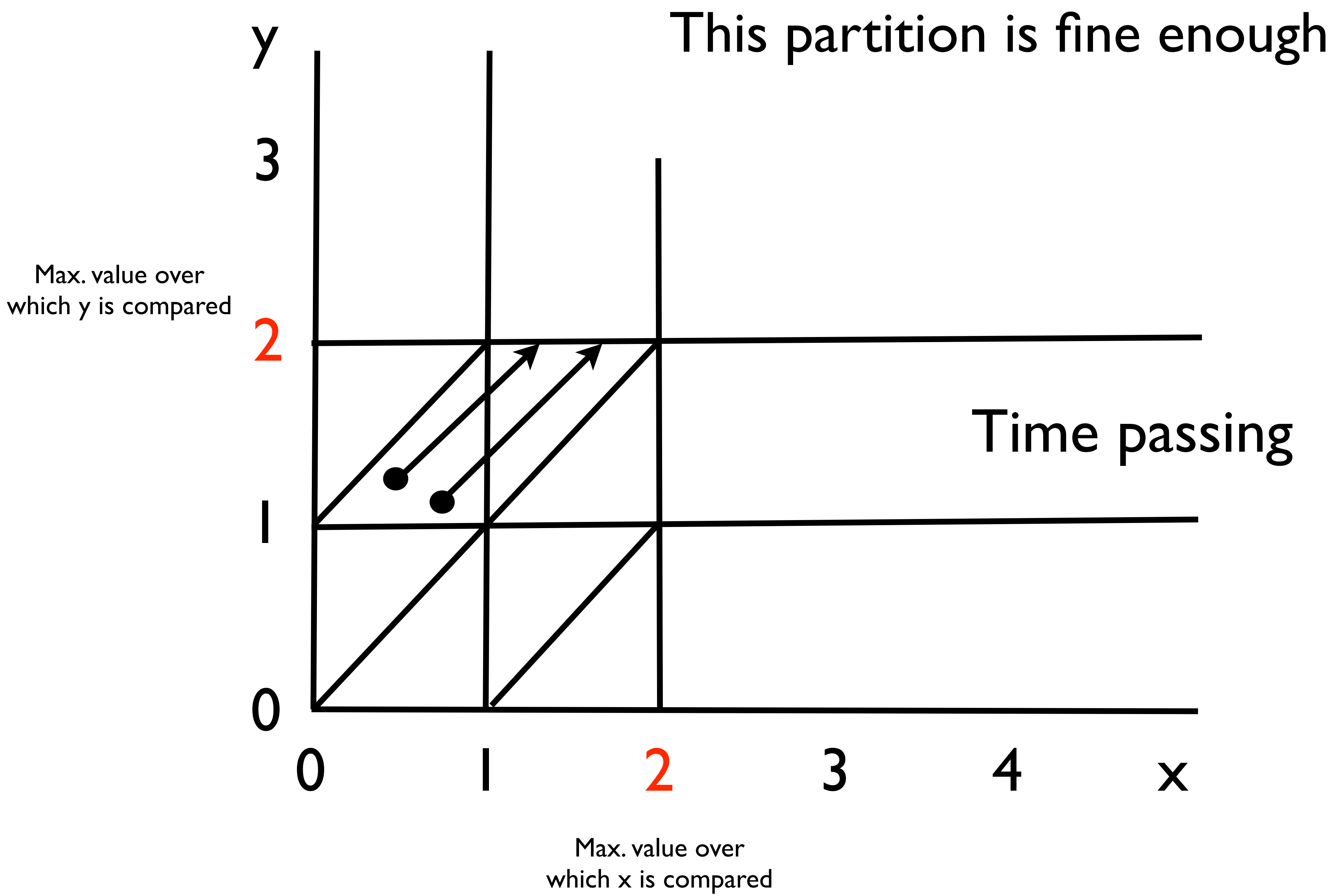
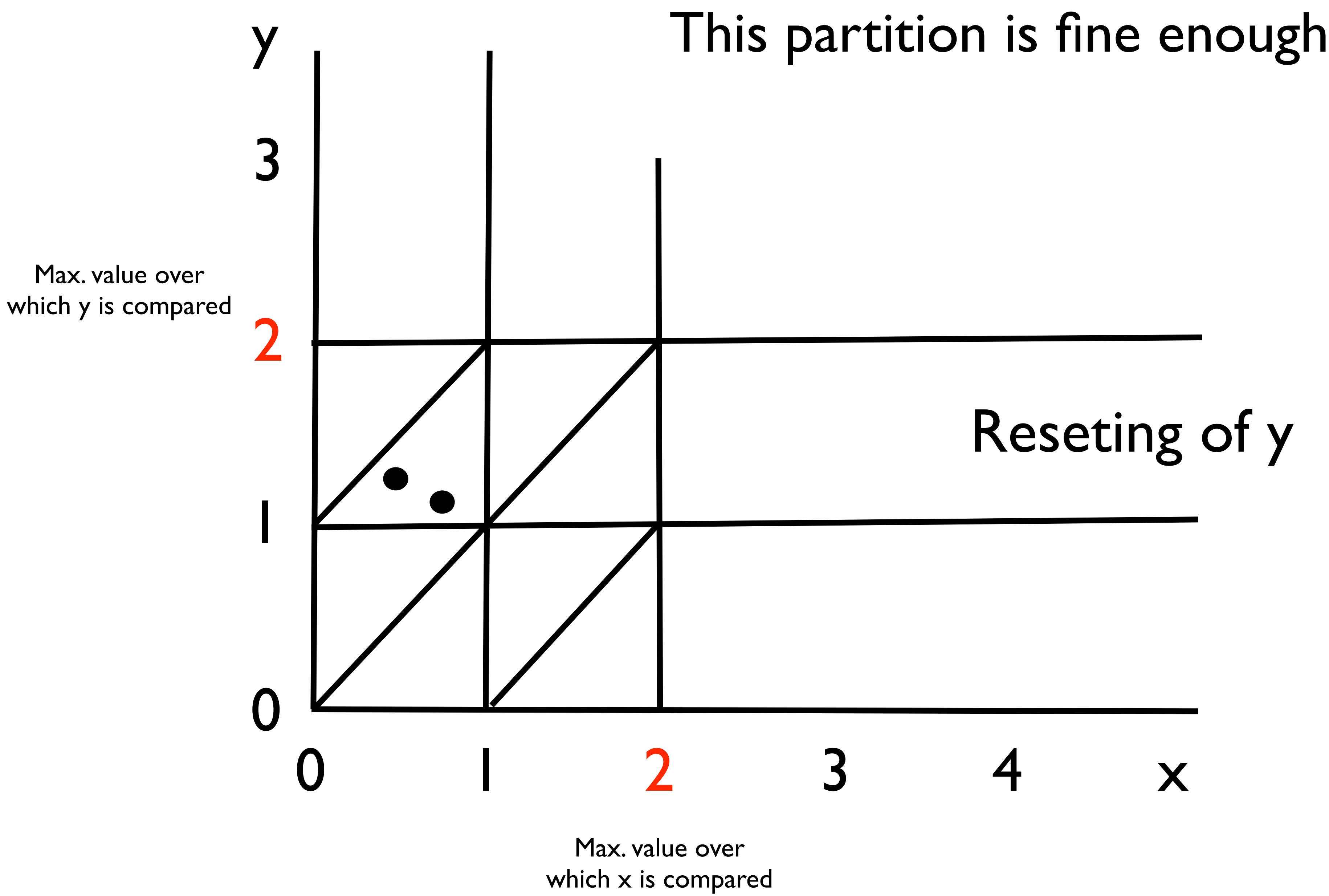
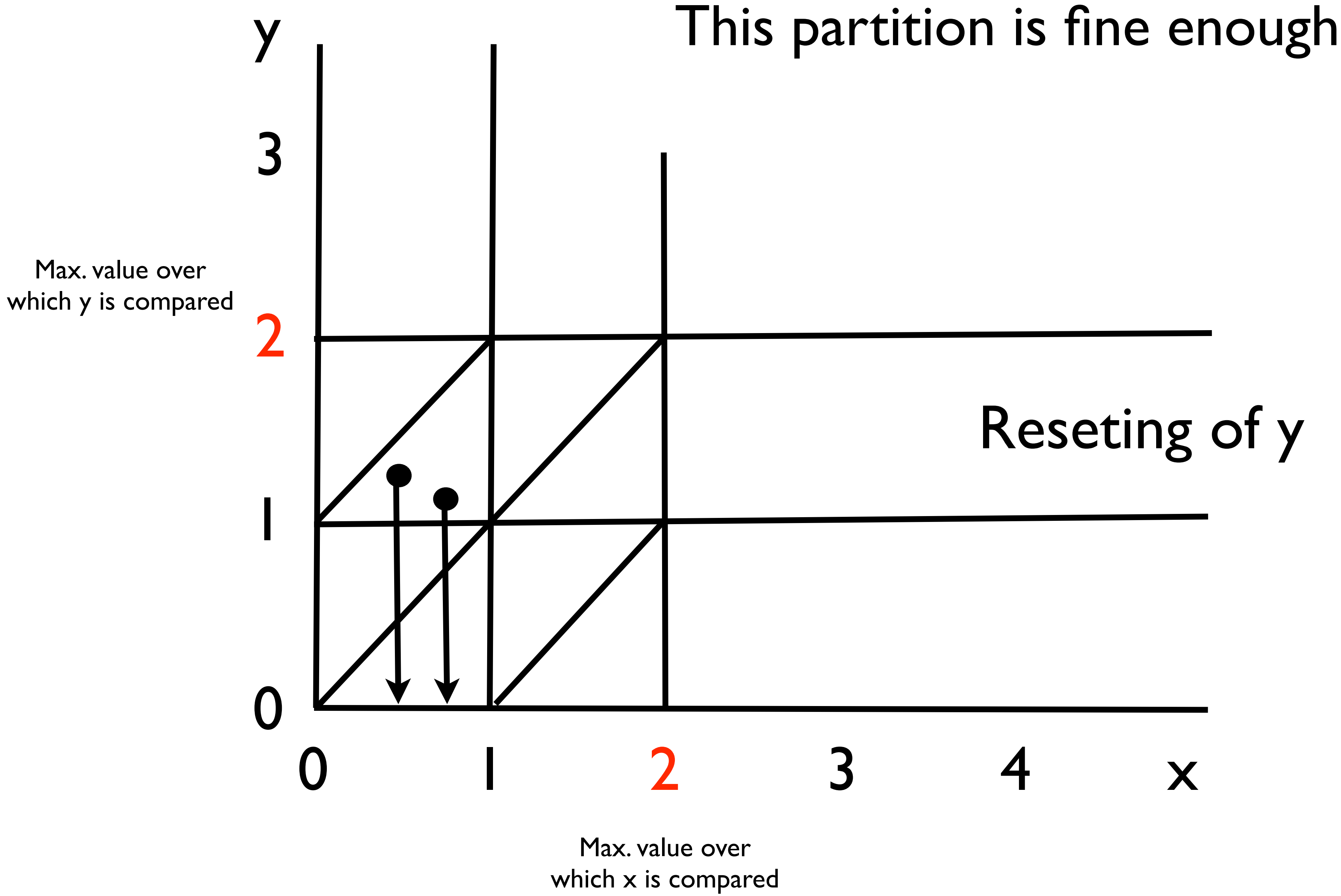


Continuous state space

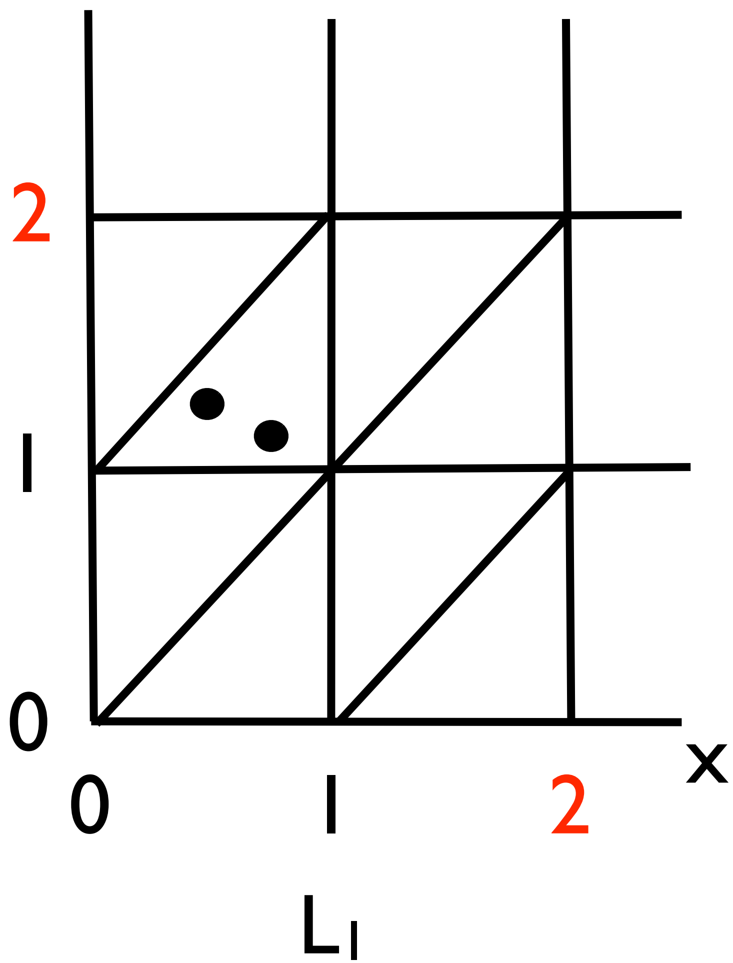


Continuous state space

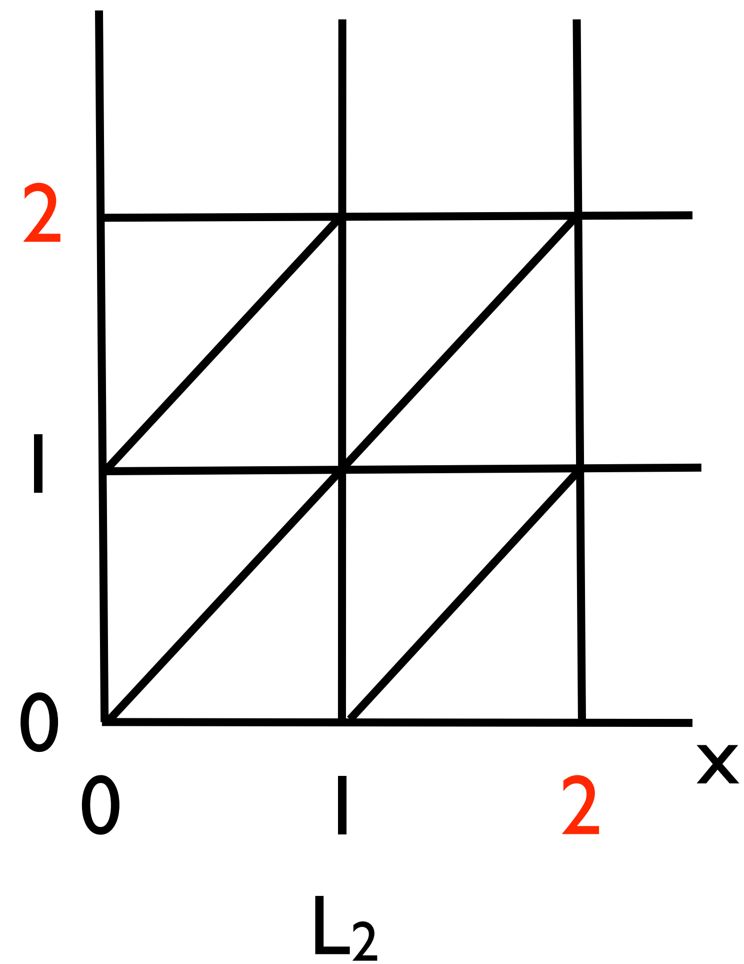


Continuous state space

This partition is fine enough



discrete transition from L_1 to L_2



Continuous state space

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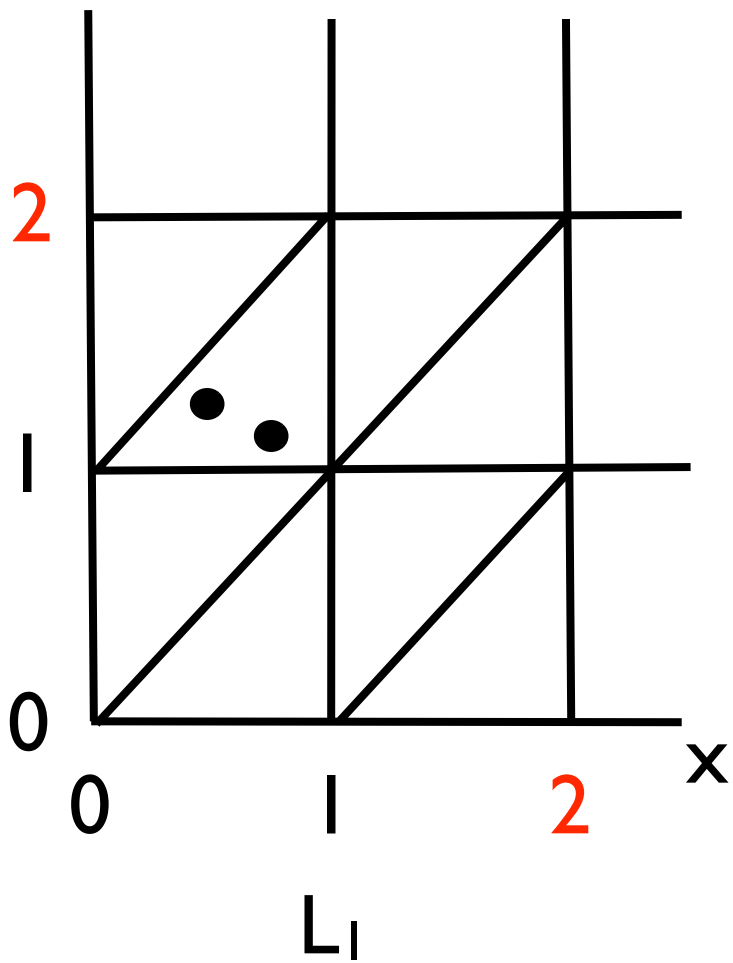
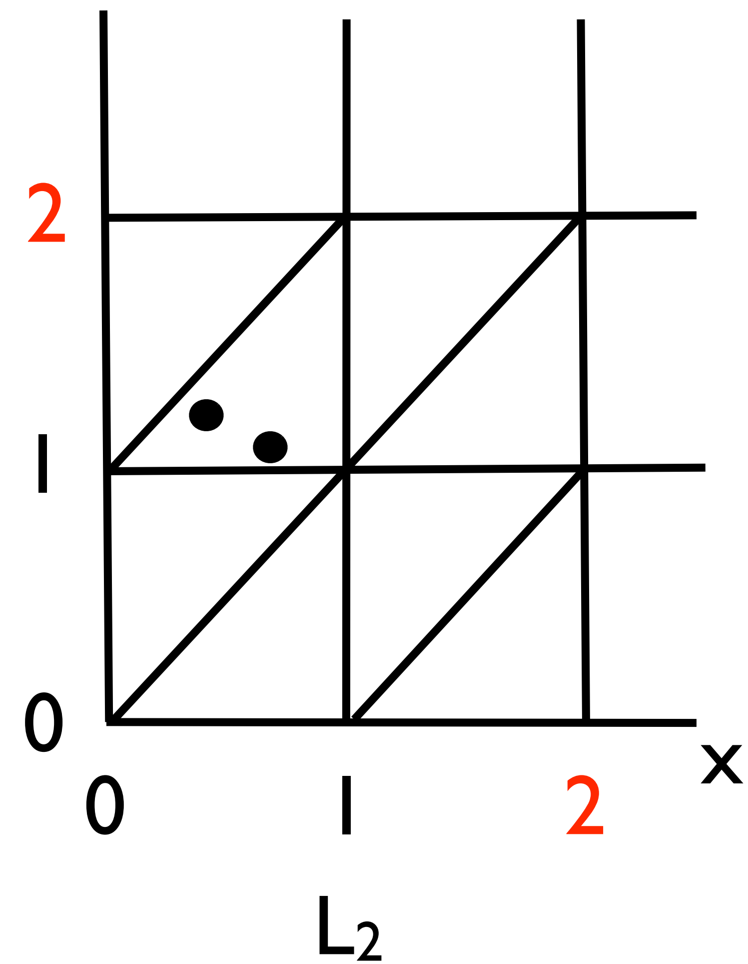
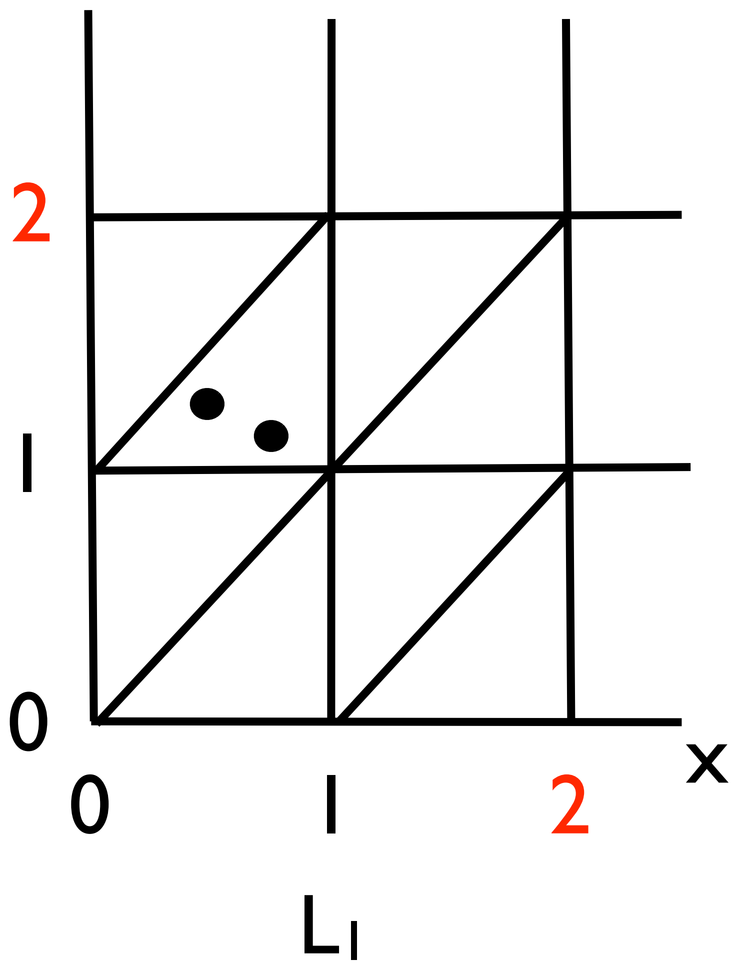


discrete transition
from L_1 to L_2

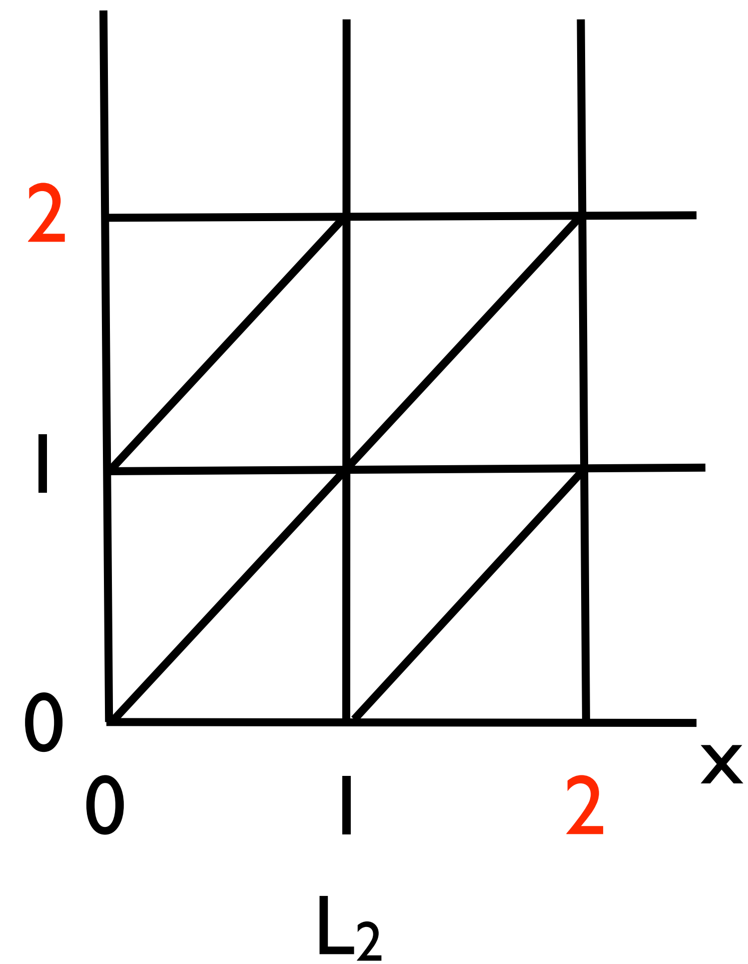


Continuous state space

This partition is fine enough

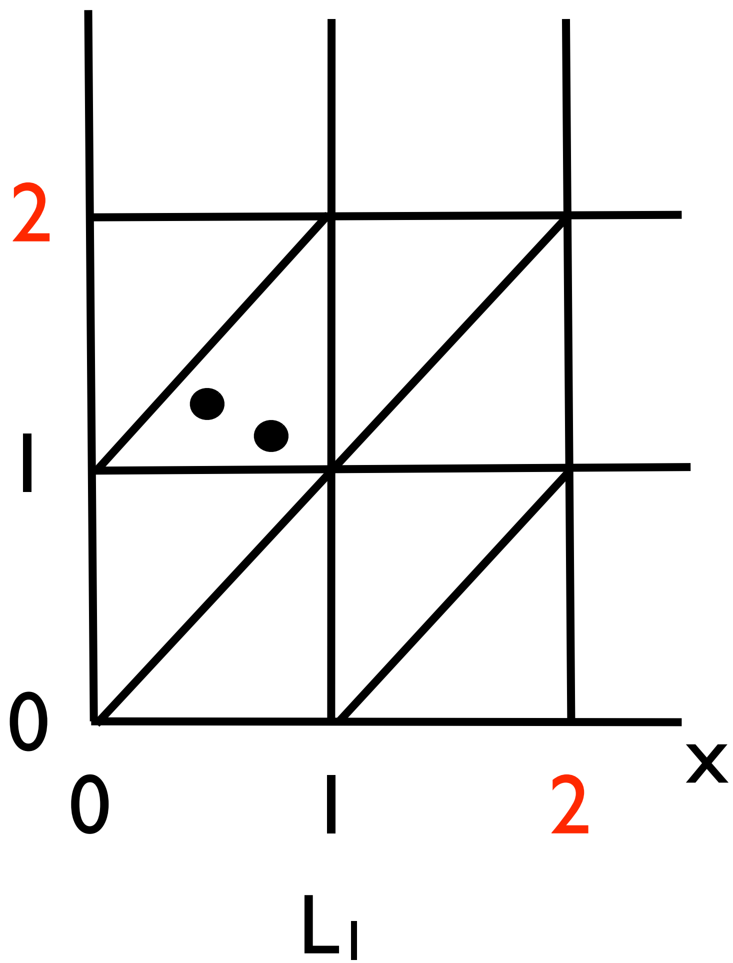


discrete transition from L_1 to L_2 with reset of y

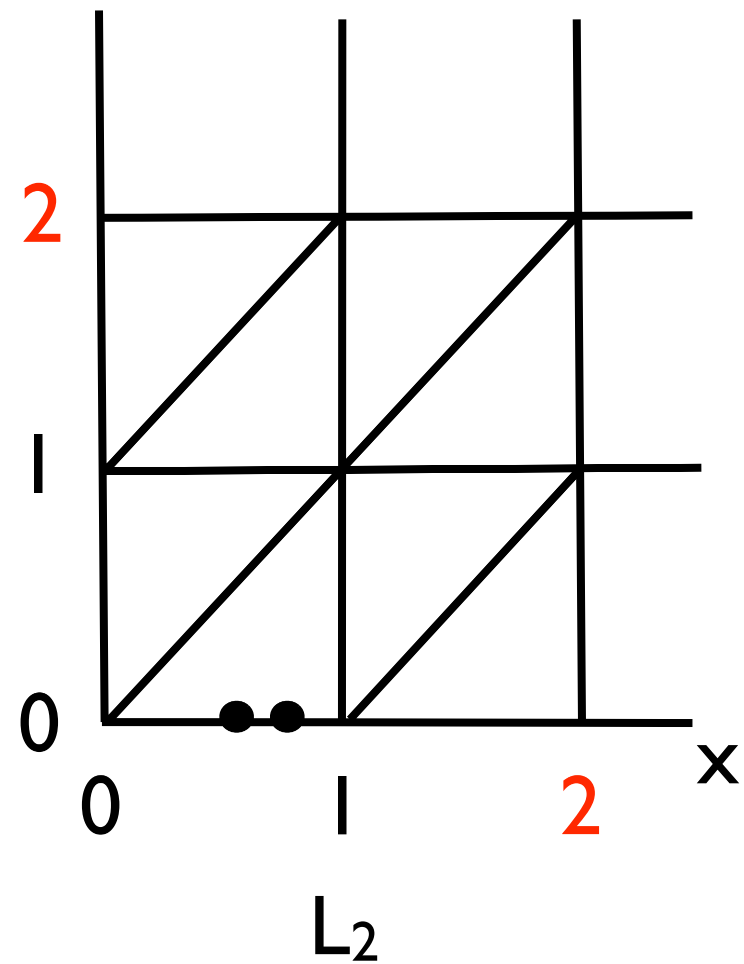


Continuous state space

This partition is fine enough



discrete transition from L_1 to L_2 with reset of y



Region equivalence: formal definition

- For each variable $x \in Cl$, let c_x be the largest constant with which x is compared in the TA.
  Two valuations $v_1, v_2 : Cl \to \mathbb{R}_{\geq 0}$ are **region equivalent**, noted $v_1 \approx v_2$ **iff**

  - same integer parts:
    for all $x \in Cl$, $\textbf{int}(v_1(x)) = \textbf{int}(v_2(x))$, or $v_1(x) > c_x$ and $v_2(x) > c_x$.

  - same fractional ordering:
    for all $x,y \in Cl$ with $v_1(x) \leq c_x$ and $v_1(y) \leq c_y$,
    $\textbf{frac}(v_1(x)) \leq \textbf{frac}(v_1(y))$ iff $\textbf{frac}(v_2(x)) \leq \textbf{frac}(v_2(y))$

  - same null fractional parts:
    for all $x,y \in Cl$ with $v_1(x) \leq c_x$ and $v_1(y) \leq c_y$,
    $\textbf{frac}(v_1(x)) = 0$ iff $\textbf{frac}(v_2(x)) = 0$

- **Theorem: a Region** is a set of valuations that are **time abstract bisimilar**.

Region equivalence quotient of the time-abstract LTS

- The following theorem is the **foundation** for the automatic verification of timed automata.
- **Theorem**. Let A be a timed automaton, let L be its time-abstract labeled transition system, let $L_{\approx}$ be its quotient by the region equivalence $\approx$, then $L_{\approx}$ is **finite** and $L_{\approx}$ is **trace equivalent** to L.

Post operations in the time abstract LTS

- To construct **"region based" bisimulation quotient** of the time-abstract LTS of a TA (or to compute on it), we must be able to compute the transition relation between regions.

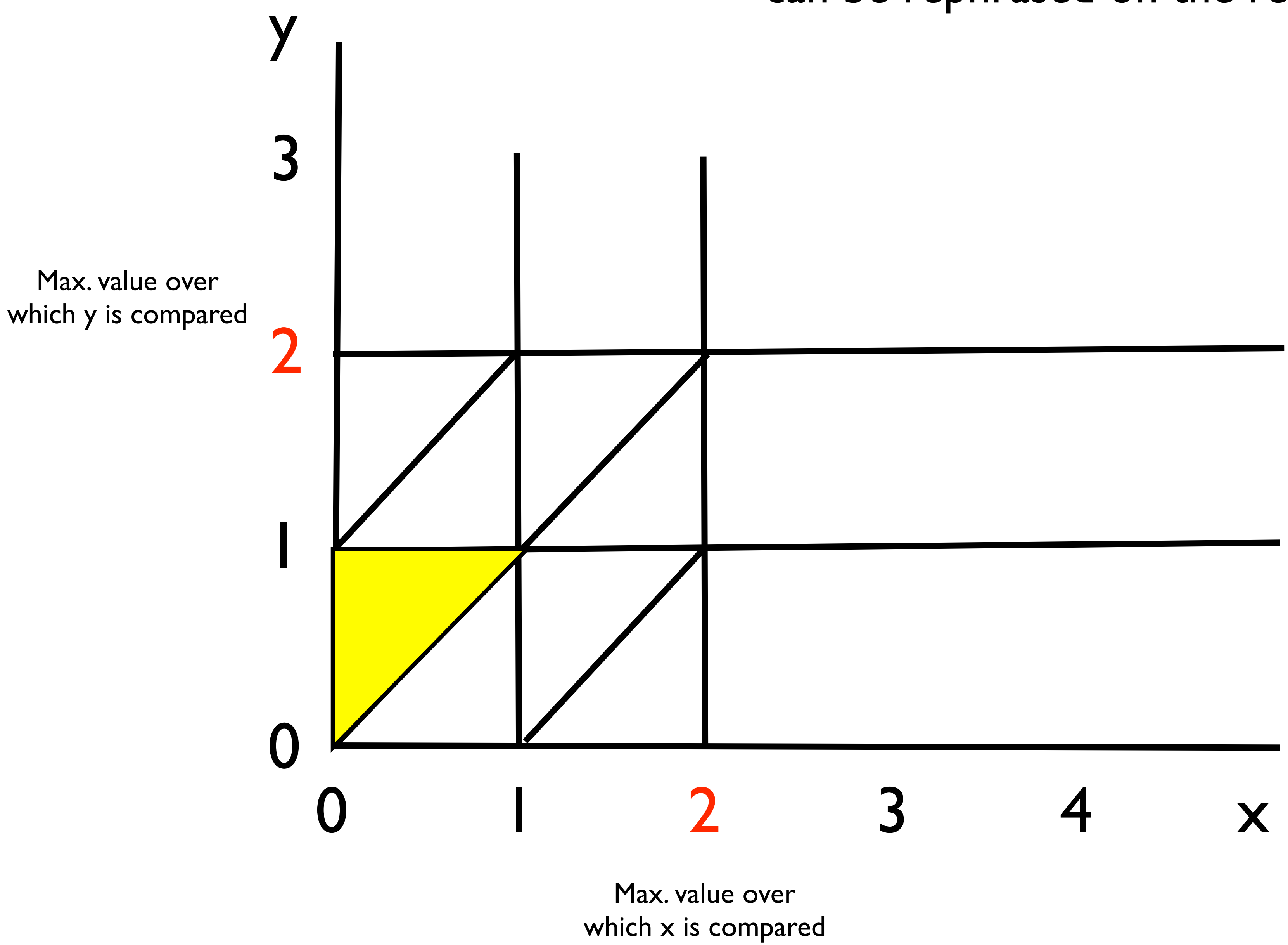
- We consider the two types of transitions that we find in the time-abstract LTS of a TA:

  - **Discrete transitions** that are associated to transition edges in the timed automaton. Let $(q_1,a,\Phi,\Delta,q_2) \in E$:
    (1) Note that given a region r and a guard Φ, all valuations $v_1,v_2 \in r$ is such that $v_1 \models \Phi$ iff $v_2 \models \Phi$.
    (2) The effect of resetting a clock on a region r gives a region r'.

  - **Delay transitions**. Given a region r, we can compute the set of regions r' that contains $v+t$ for some $v \in r$ and some $t \in \mathbb{R}$.

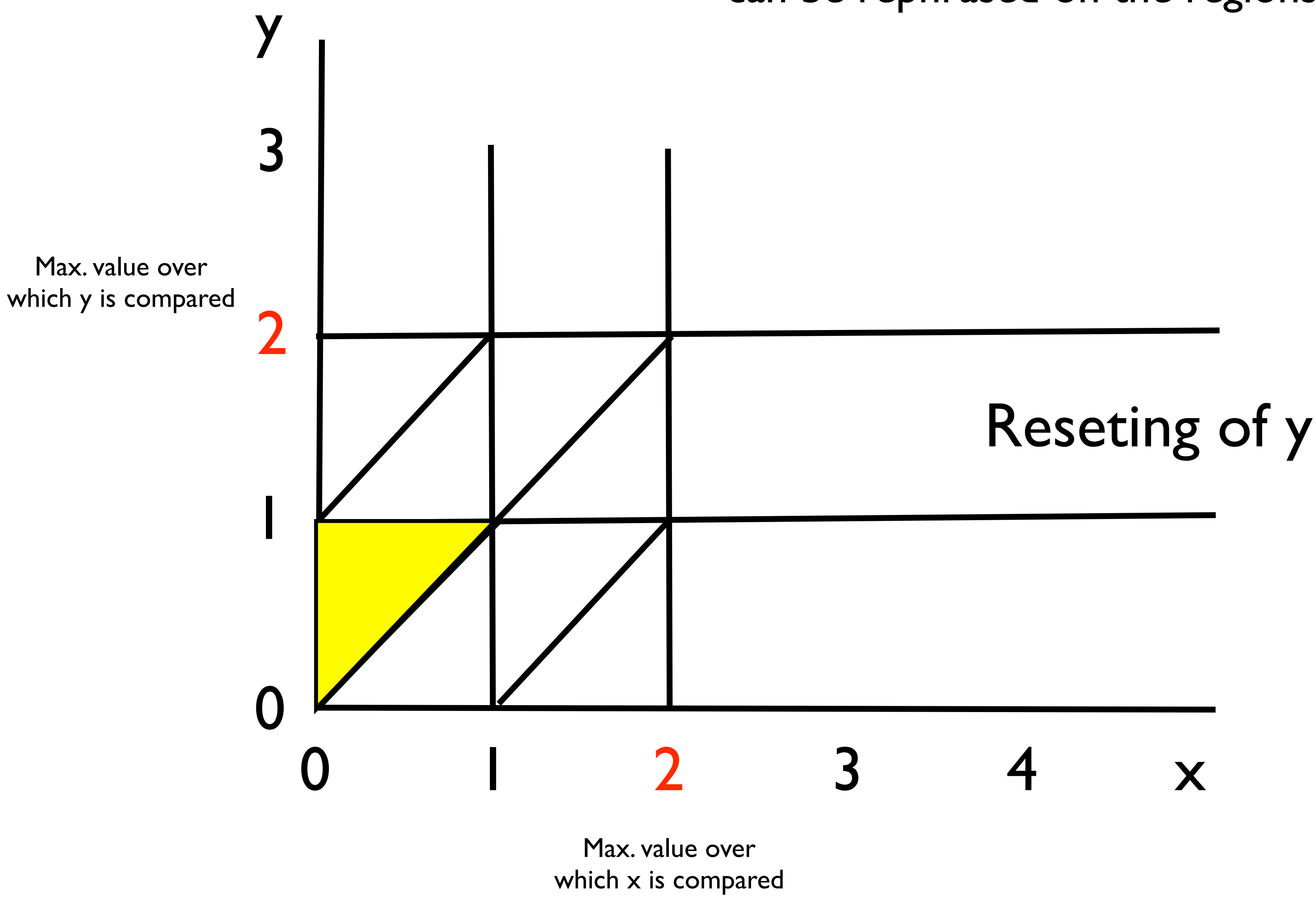
Continuous state space

The transitions of the time-abstract transition system can be rephrased on the regions



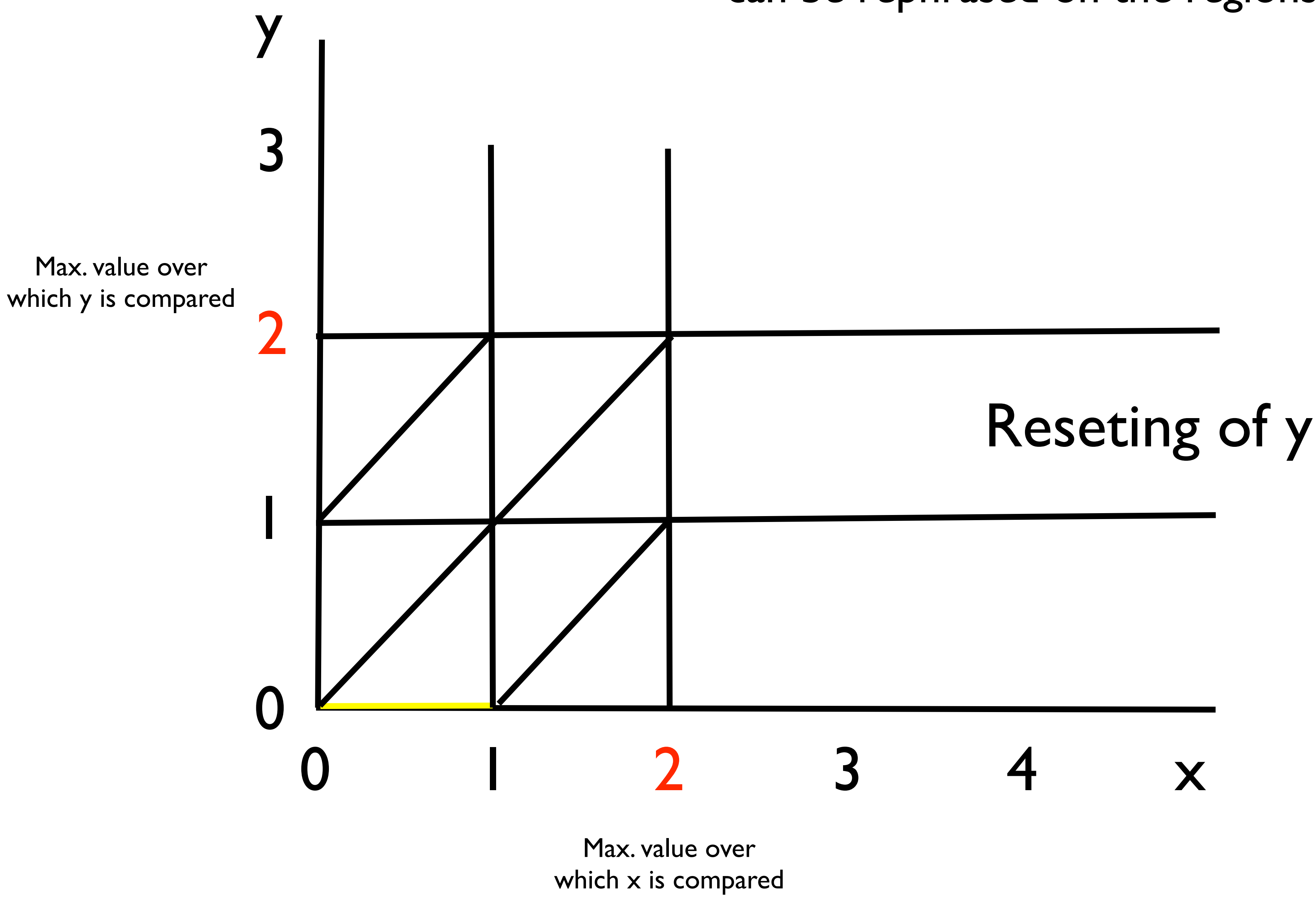
Continuous state space

The transitions of the time-abstract transition system can be rephrased on the regions



Continuous state space

The transitions of the time-abstract transition system can be rephrased on the regions



Continuous state space

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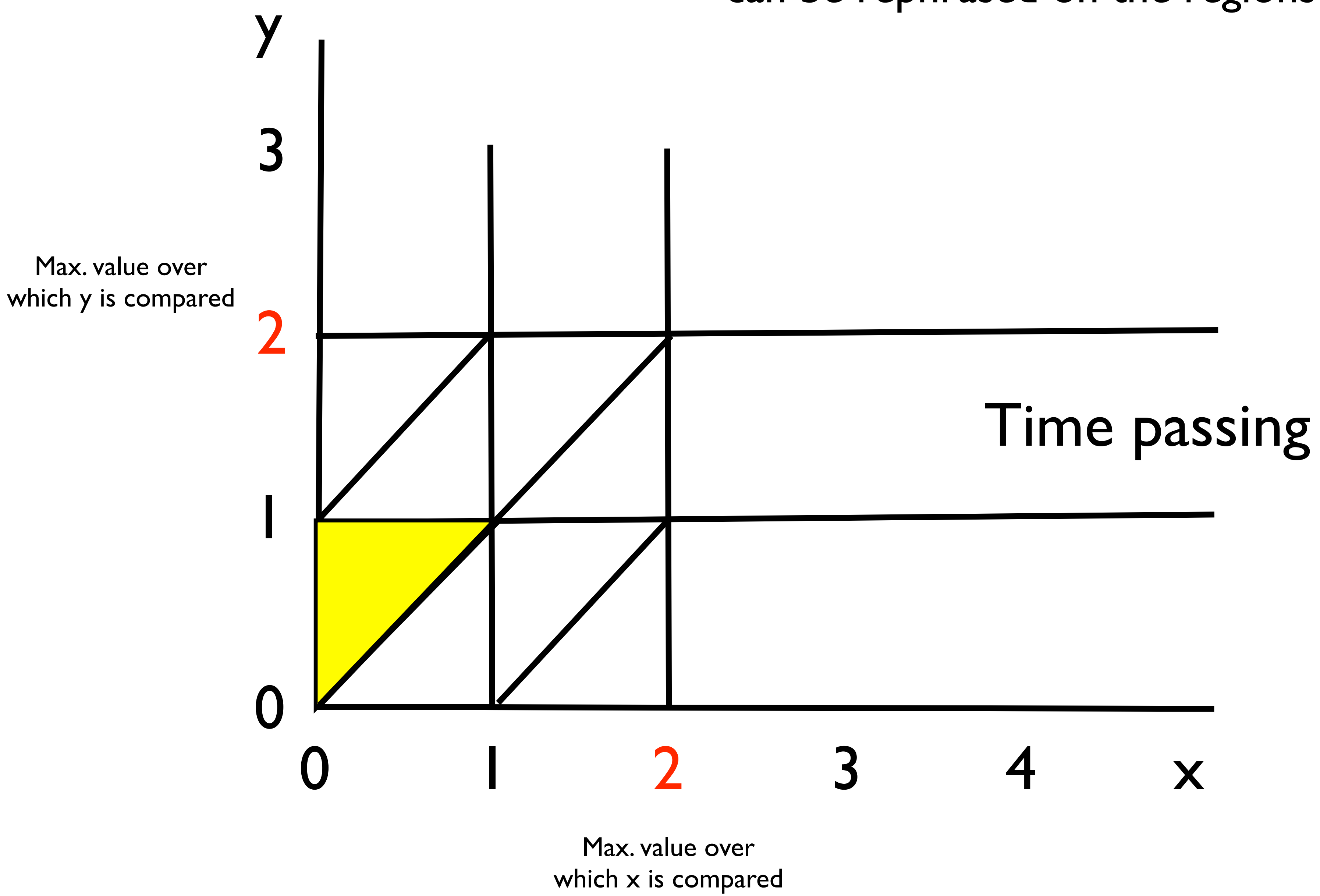
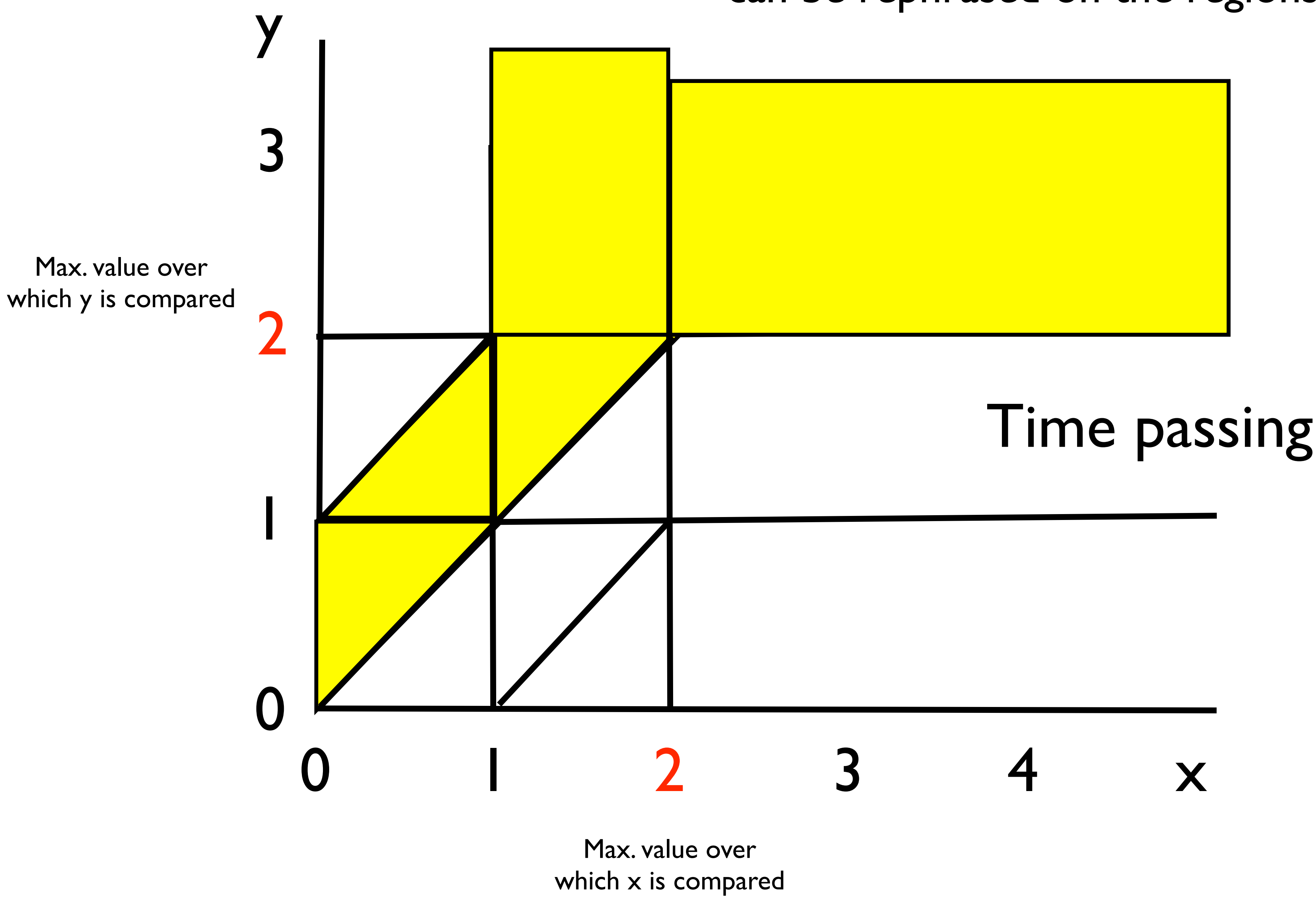


Continuous state space

The transitions of the time-abstract transition system can be rephrased on the regions

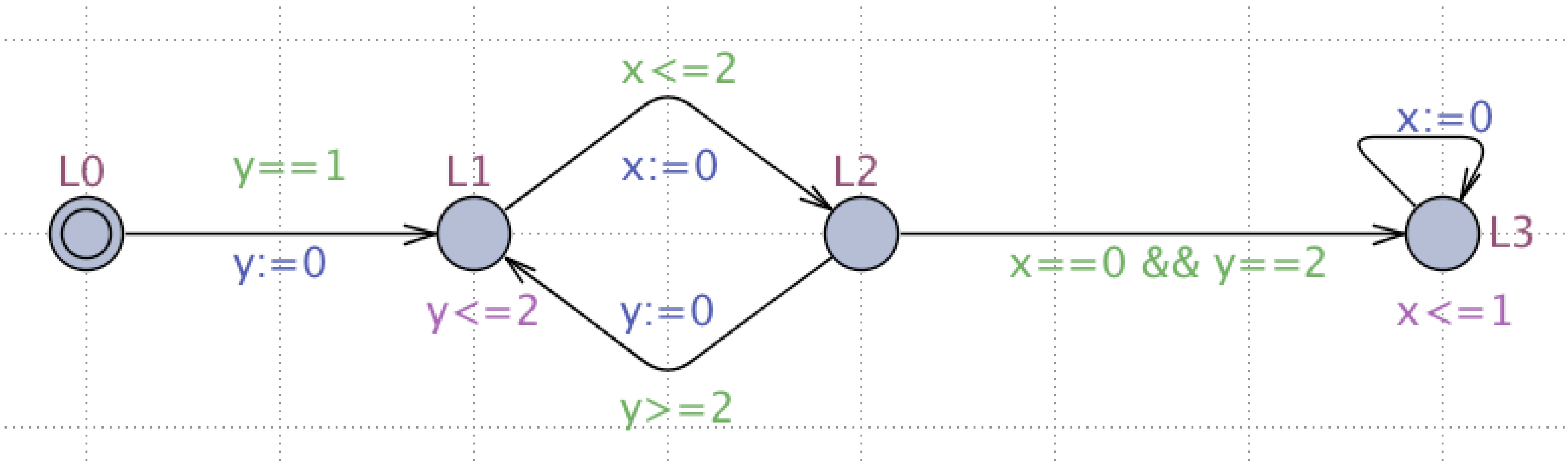


TA and the backward approach

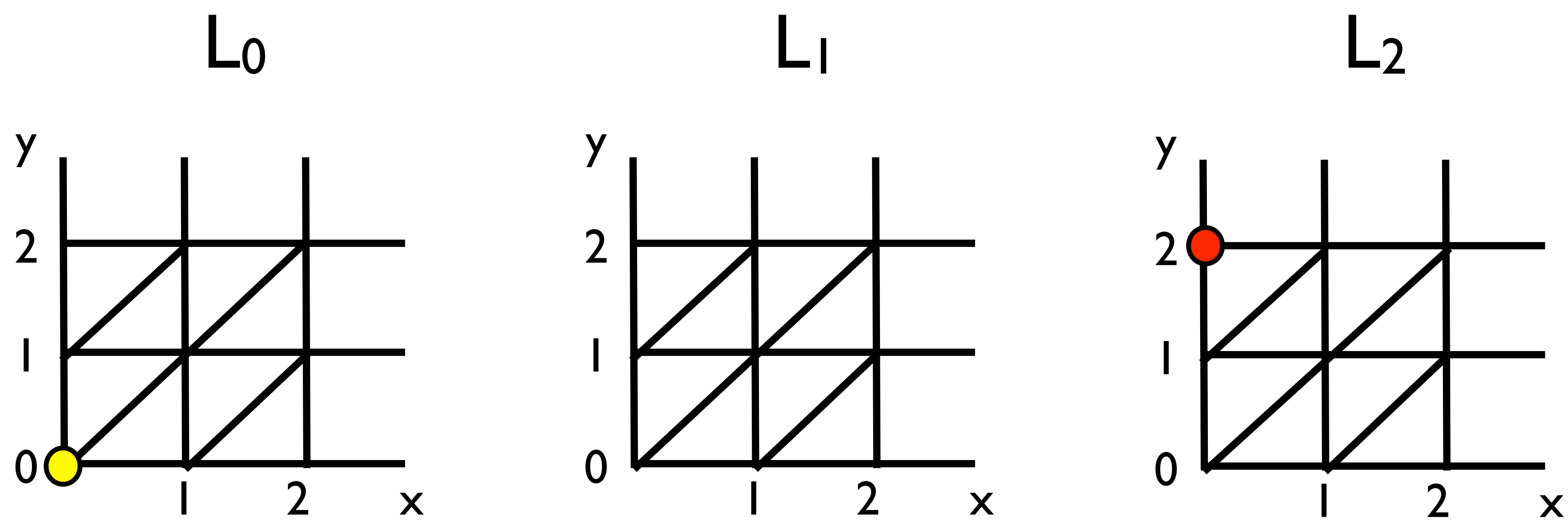
- The **region bisimulation** is **stable** also for the operations that explore the state space in a **backward fashion** (Pre, Apre)
- ... and so, we can also use backward algorithms to verify TA.

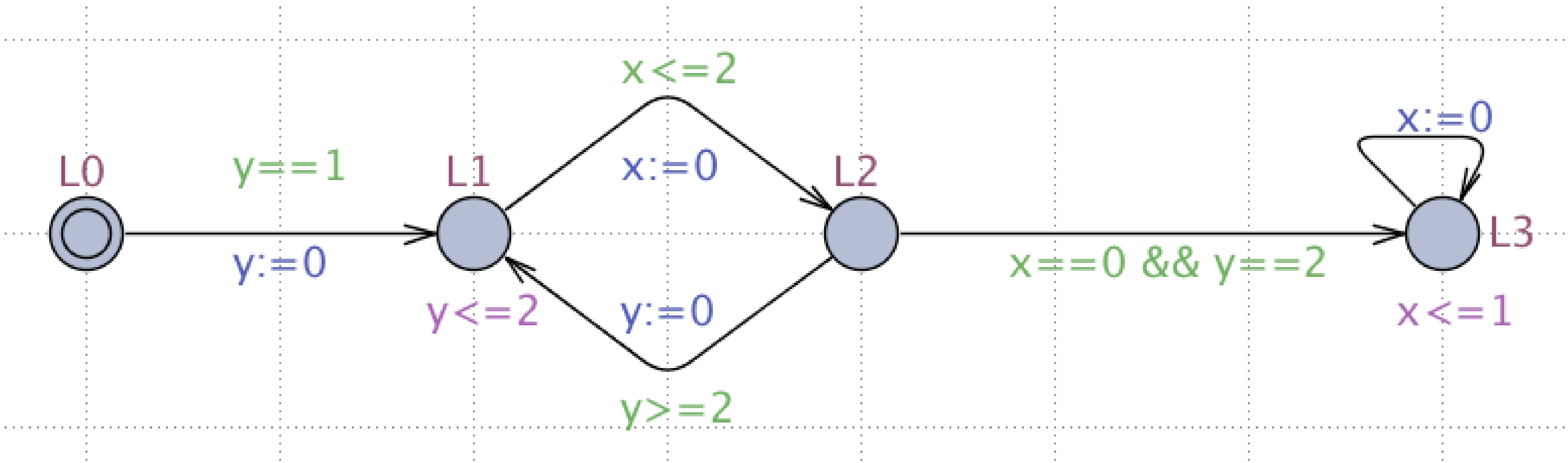
On the use of the region equivalence to verify reachability properties of TA

Forward reachability analysis on a simple example



Question: Can L3 be reached ?





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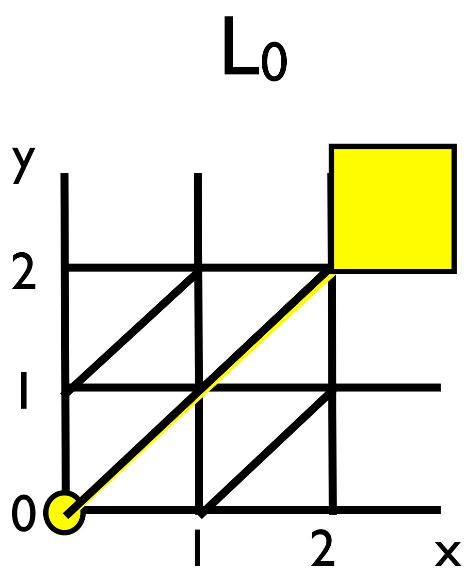

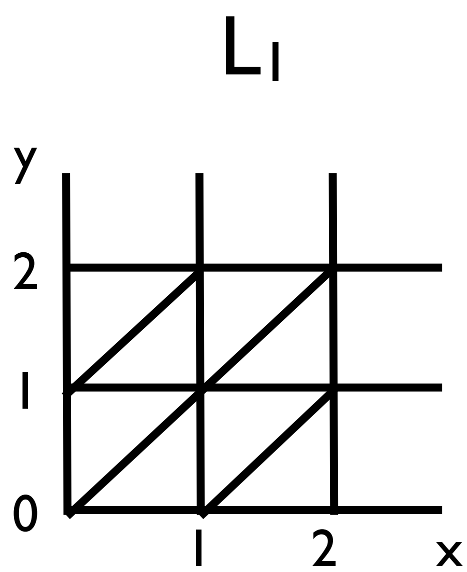

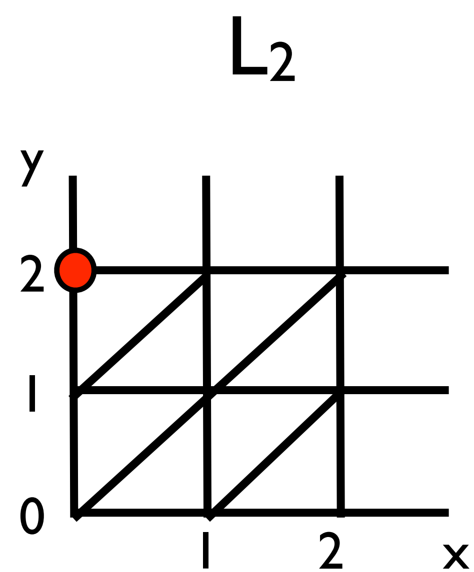

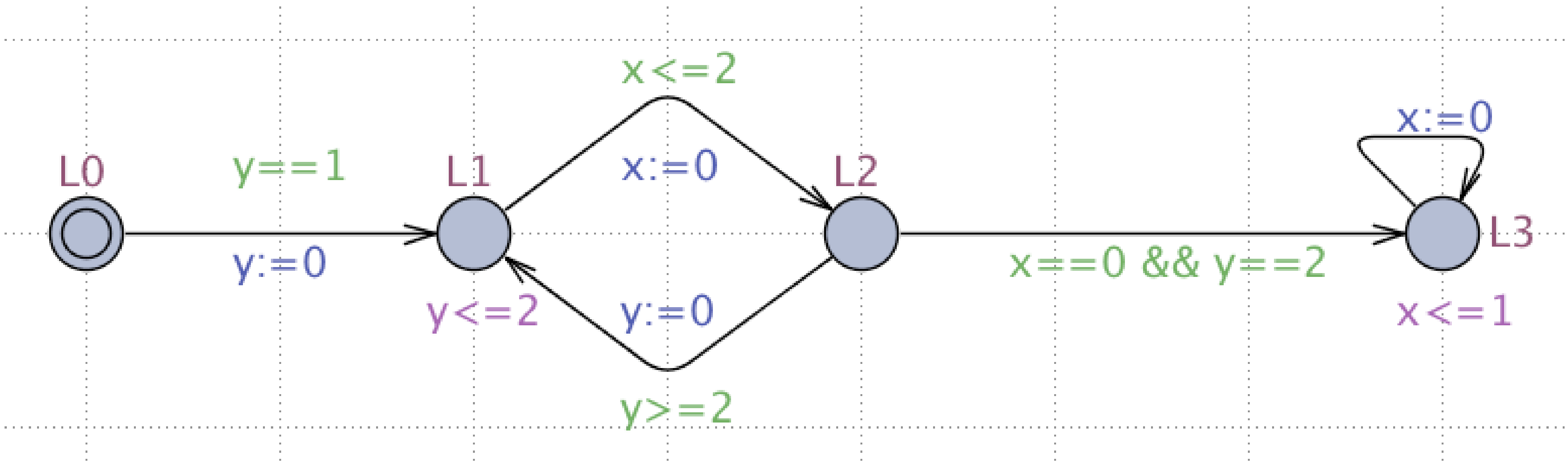

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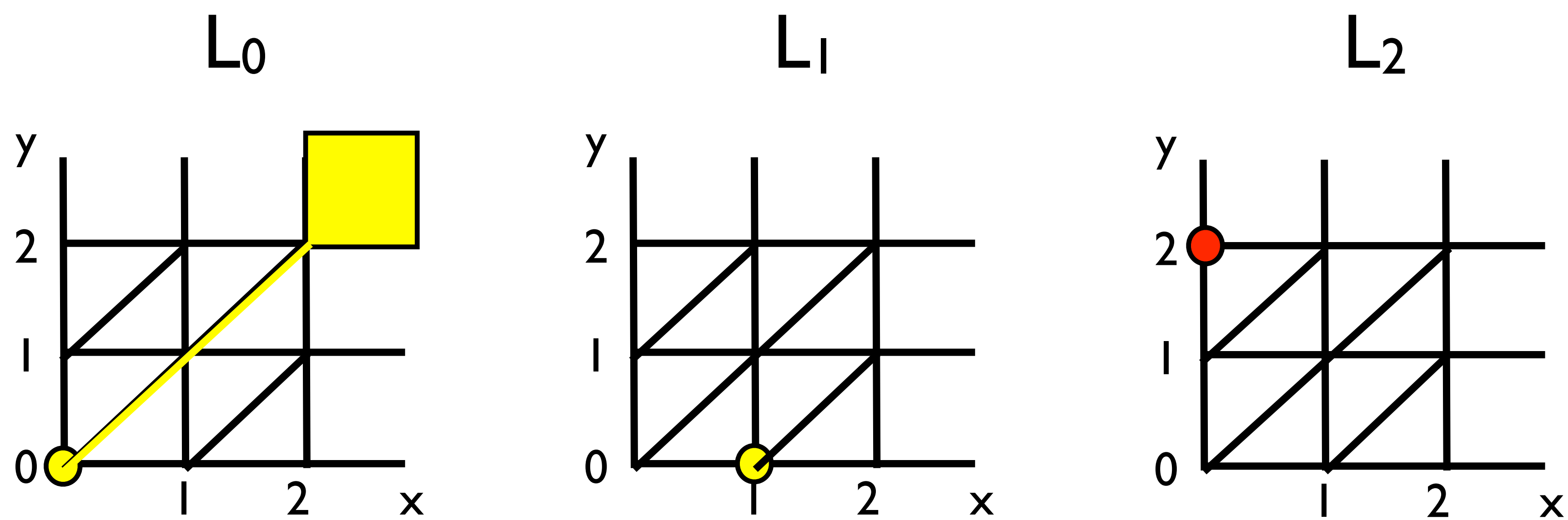

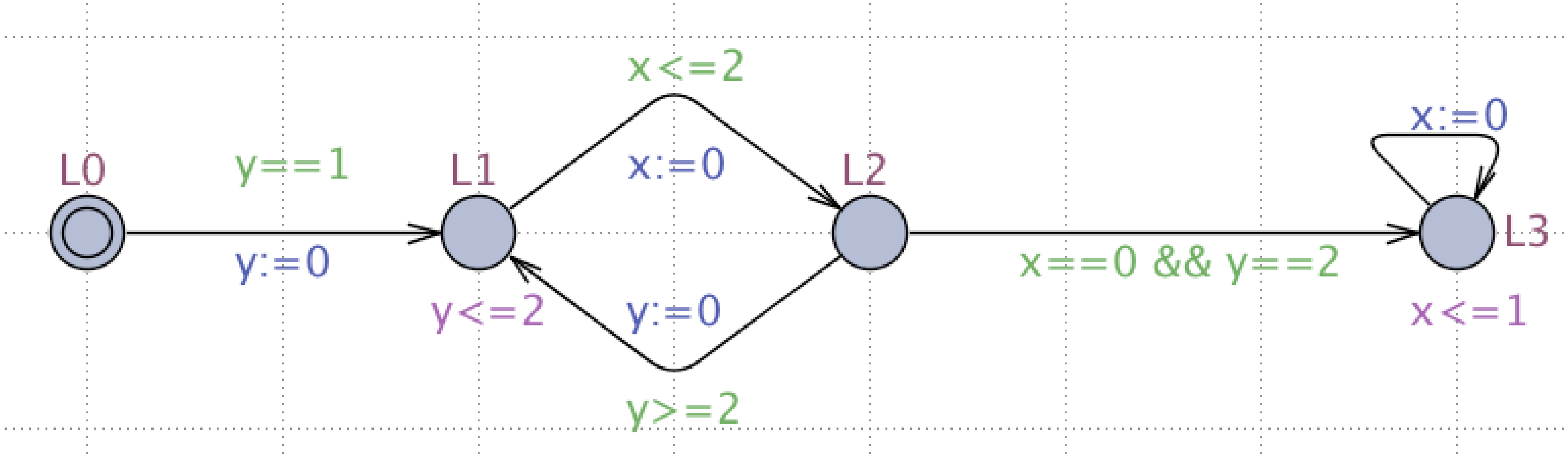

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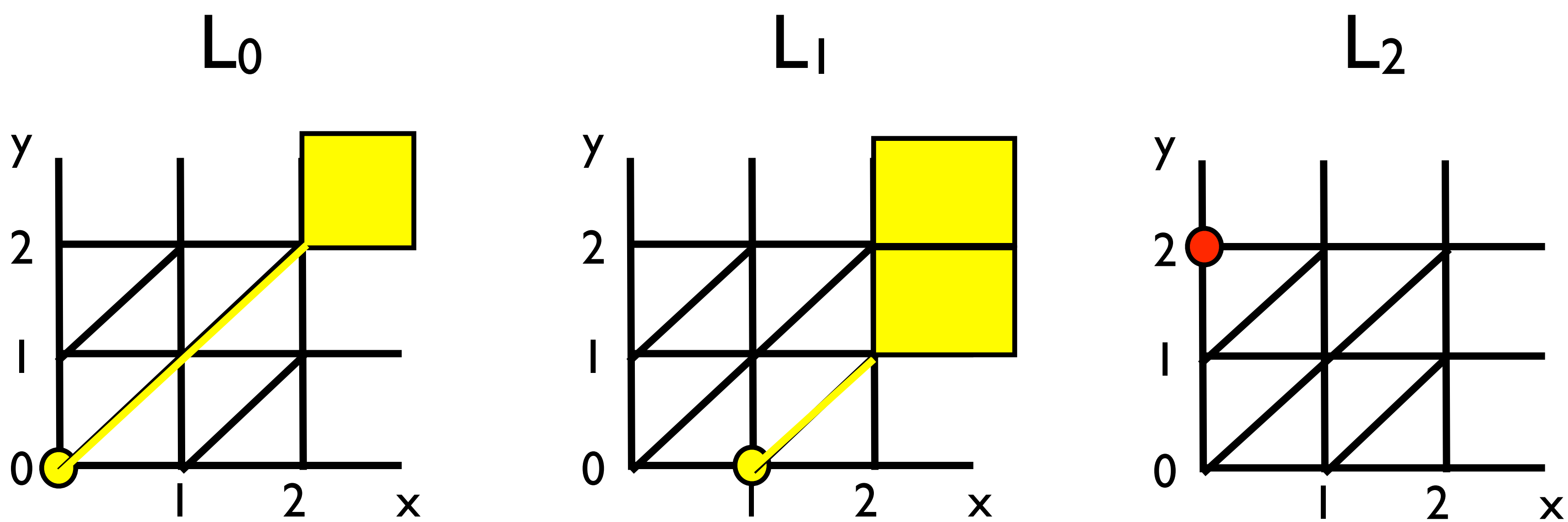

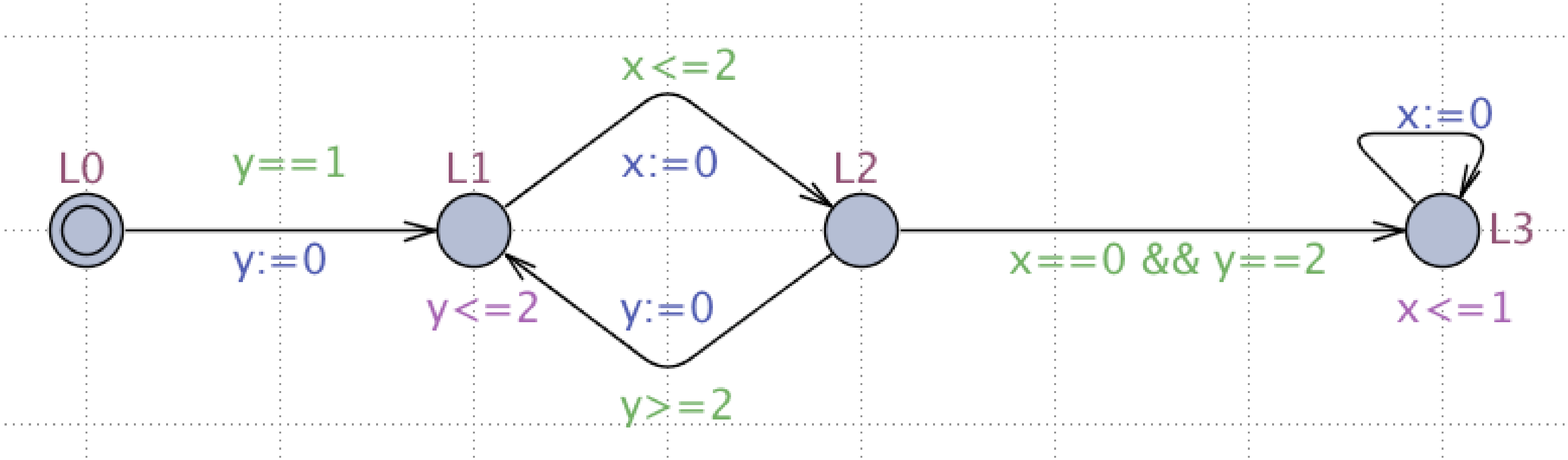

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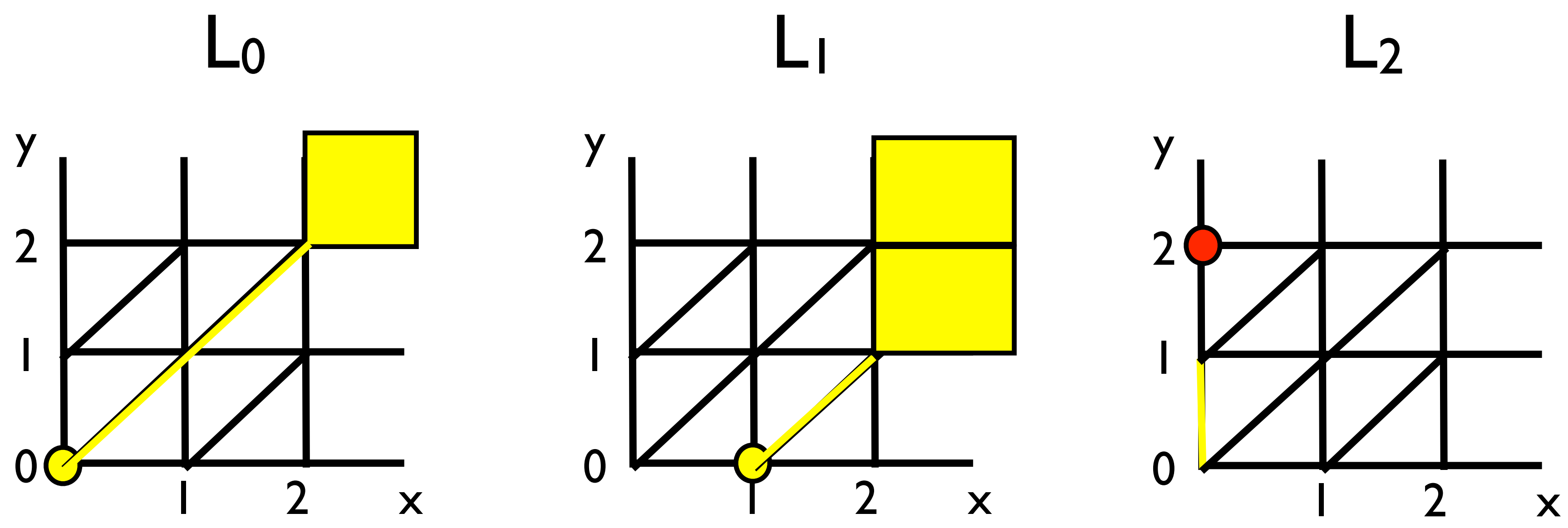

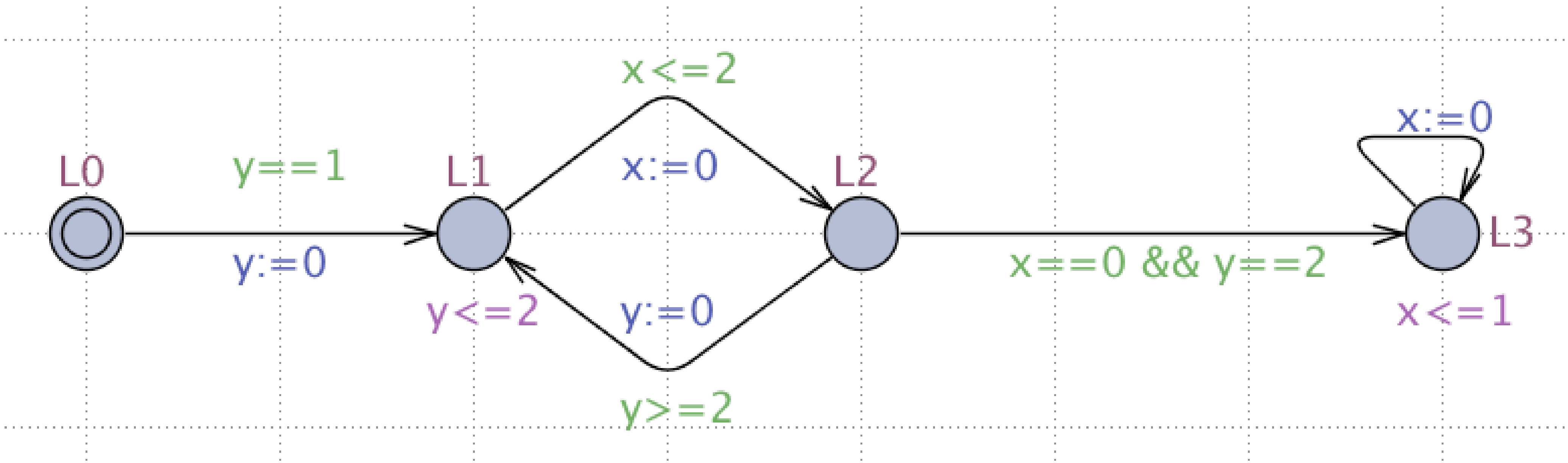

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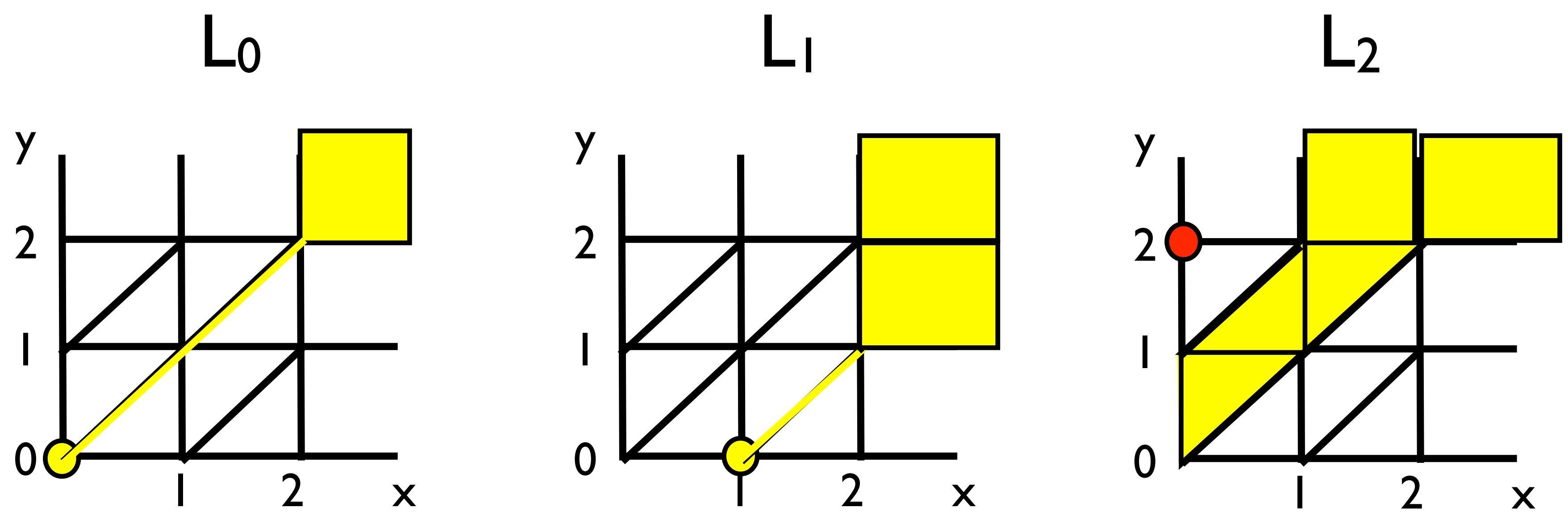

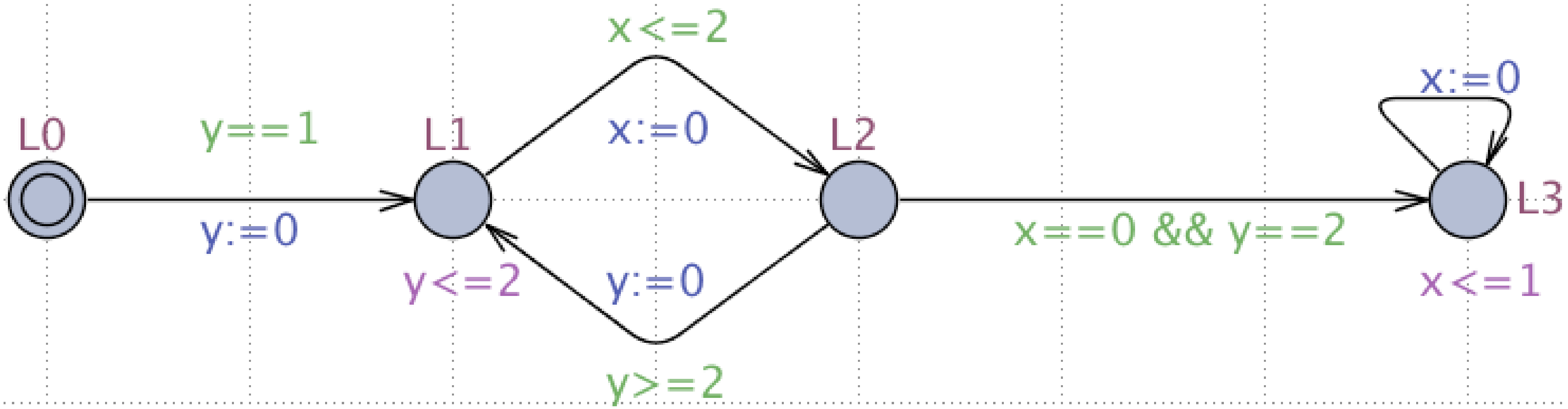

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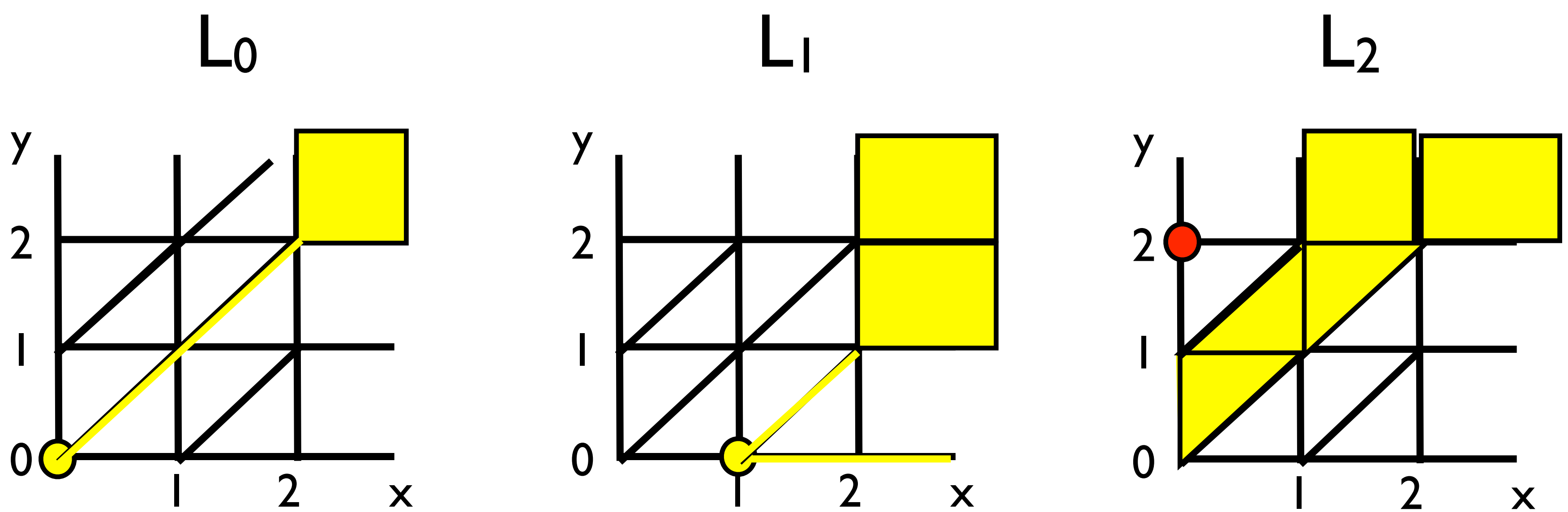

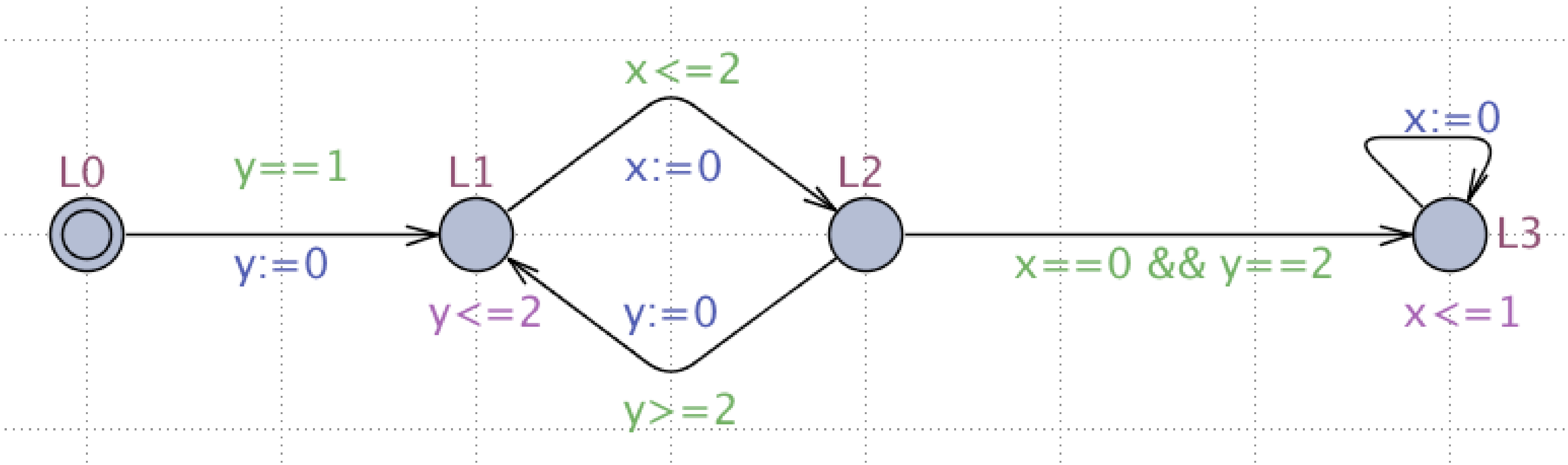

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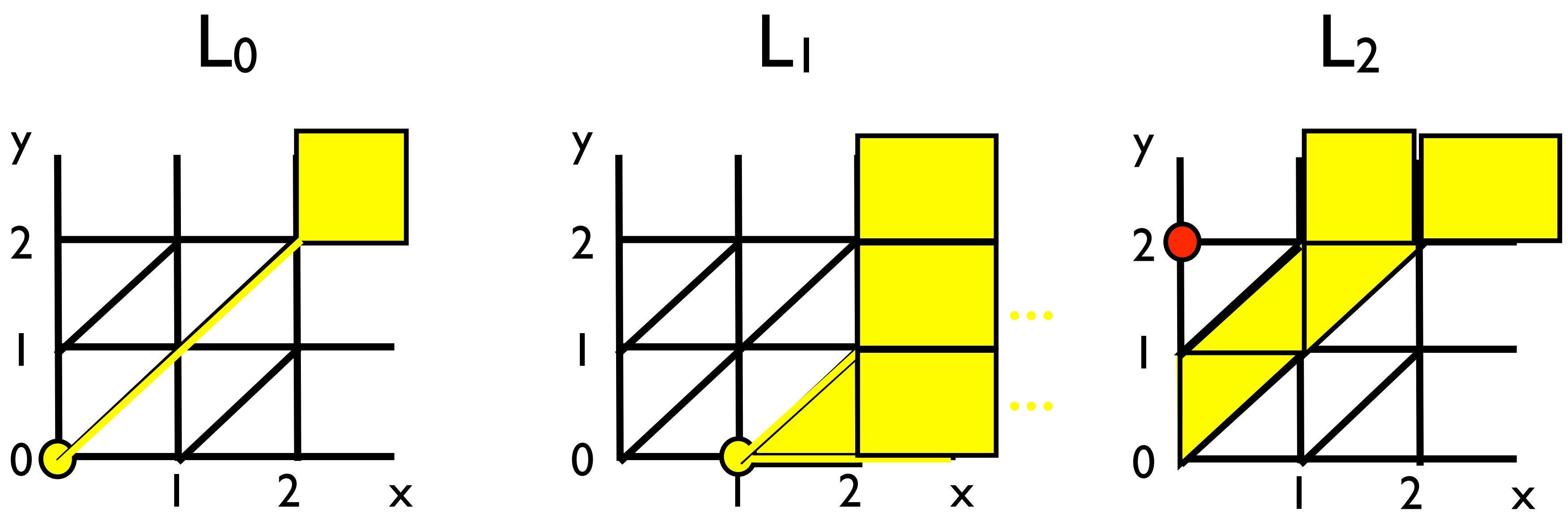

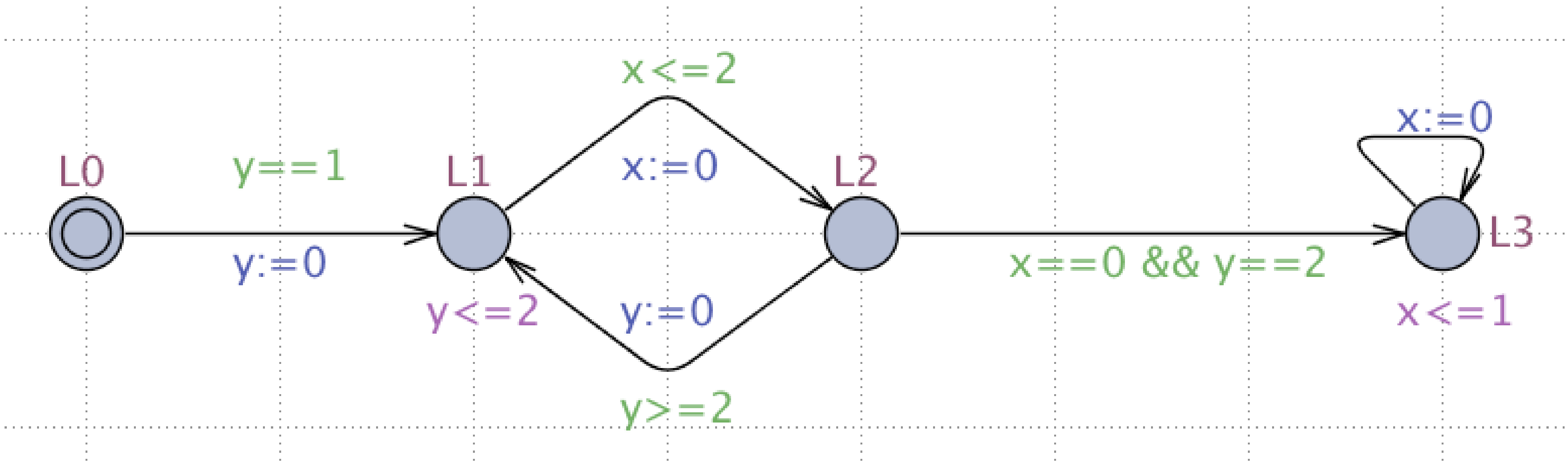

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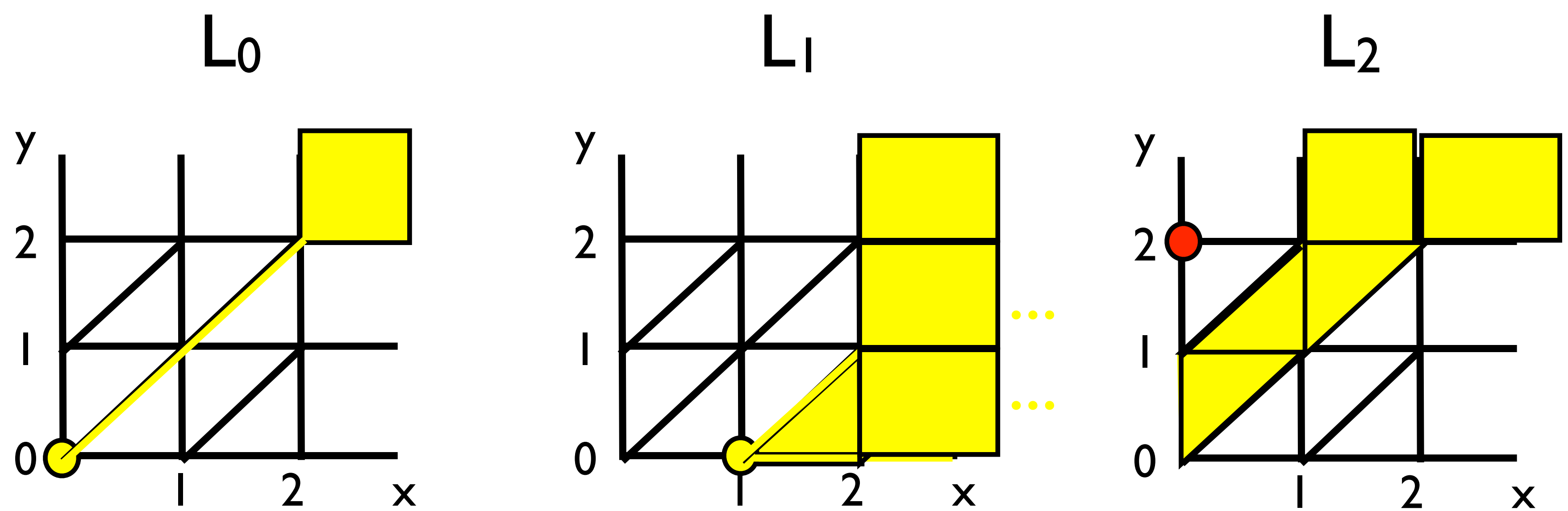
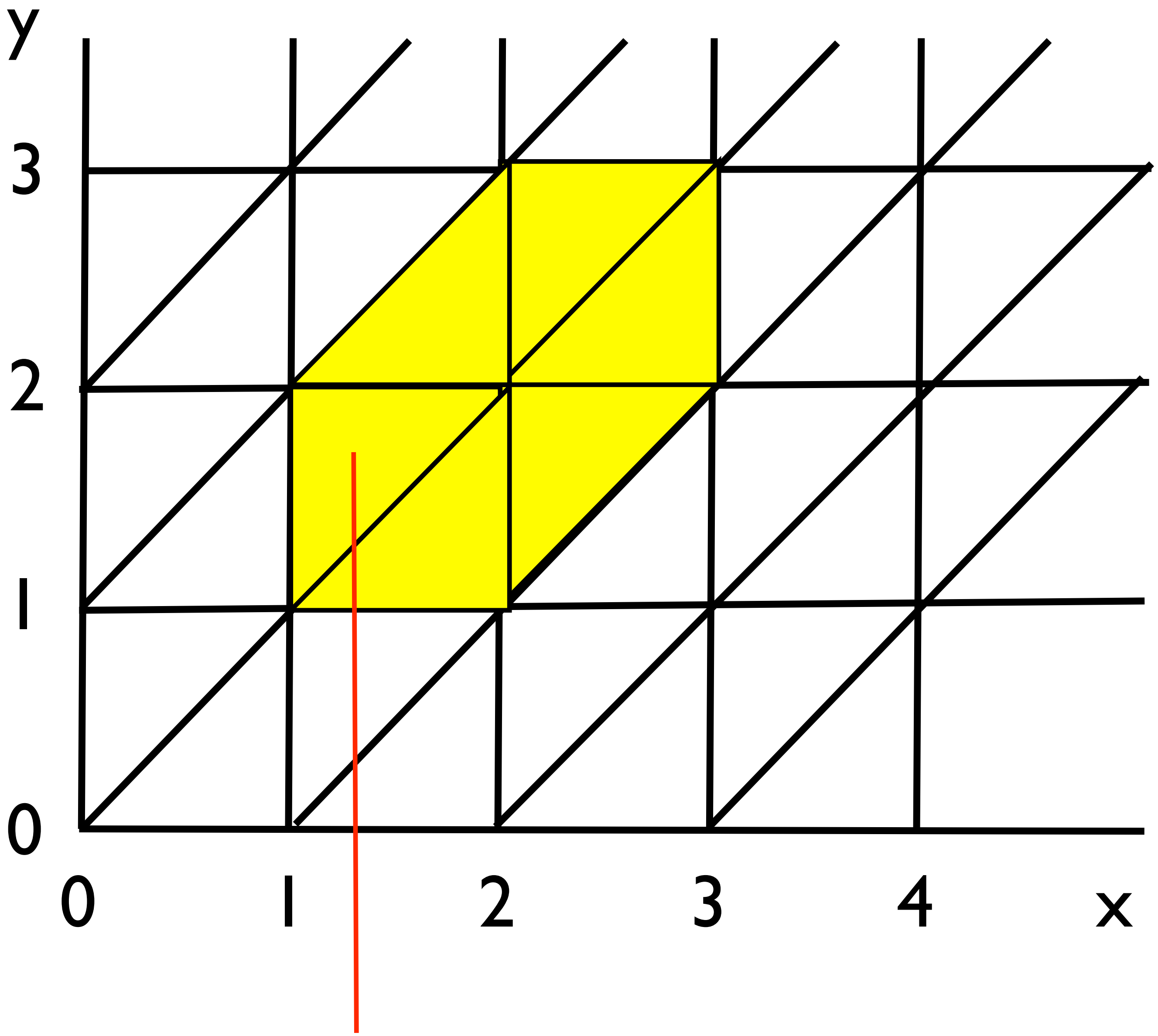


Taking the transition from L_1 to L_2 does not add any new states, so we have reached a fixed point.

Zones

- The region equivalence gives rise to a **finite quotient** and guaranties that our fixed point algorithms are **terminating**.
- Nevertheless, the number of region is **exponential** in the **number of clocks** as well as in the binary encoding of **constants**.
- To mitigate this state explosion phenomenon, we can use efficient data-structure to represent **convex unions of regions**. **Zones** are such a data-structure.
- Note that the reachability problem for timed automata is **complete for PSpace**, so it is believed that the state explosion is unavoidable in the worst case.

Zones



$x \geq 1 \wedge x \leq 3 \wedge y \leq 3 \wedge y \geq 1 \wedge x-y \geq -2 \wedge y-x \geq 1$

Zones

- Let $\mathbb{C}$ be a finite set of clocks, a **zone** is defined by a set of constraints of the form:

  (1) $x\text{-}y \sim c$ where $x,y \in \mathbb{C}$ and $c \in \mathbb{Z}$.

  (2) $x \sim c$ where $x \in \mathbb{C}$ and $c \in \mathbb{Z}$.

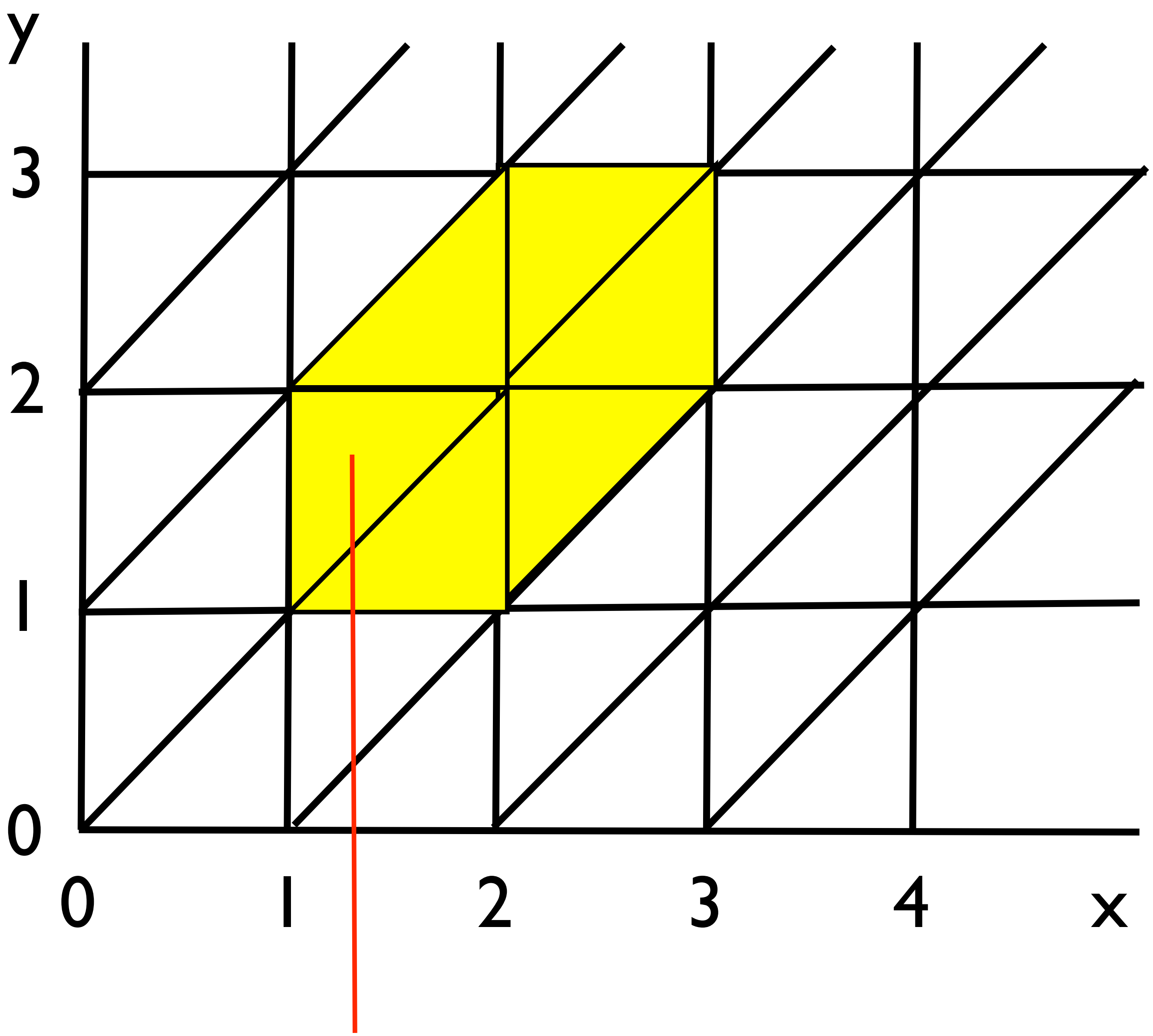
  and $\sim \in \{ \leq,<,=,>,\geq \}$.

- **Zones** are closed under the **reseting operation**, the **forward** and **backward time passing** operations, and intersection.

- Unfortunately, zones are **not** closed under union nor complementation. So implementations need to maintain lists of zones.

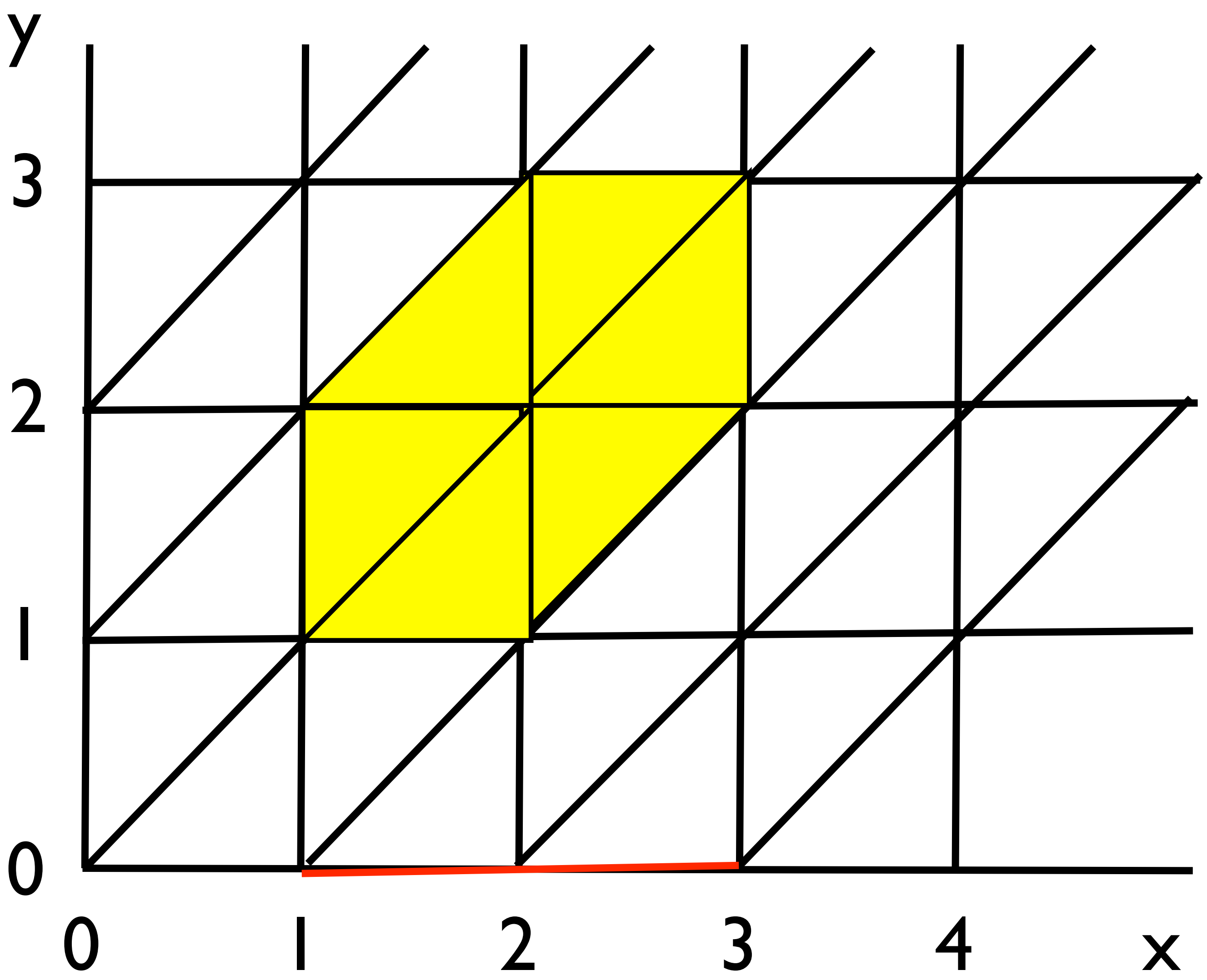
- Zones can be canonically represented by **DBM** (=Difference Bound Matrices).

Zones



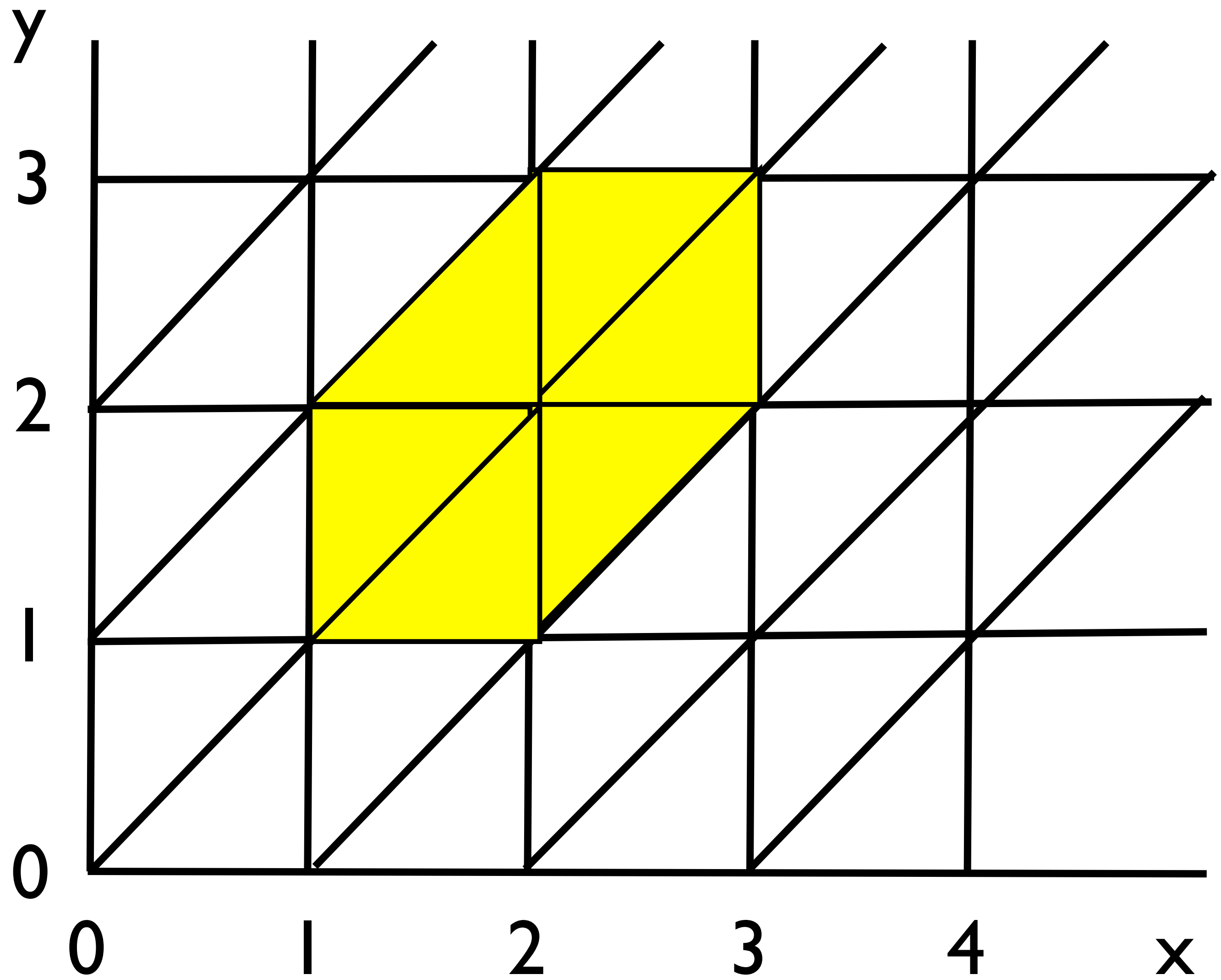
$x \geq 1 \wedge x \leq 3 \wedge y \leq 3 \wedge y \geq 1 \wedge x - y \geq -2 \wedge y - x \geq 1$

Zones



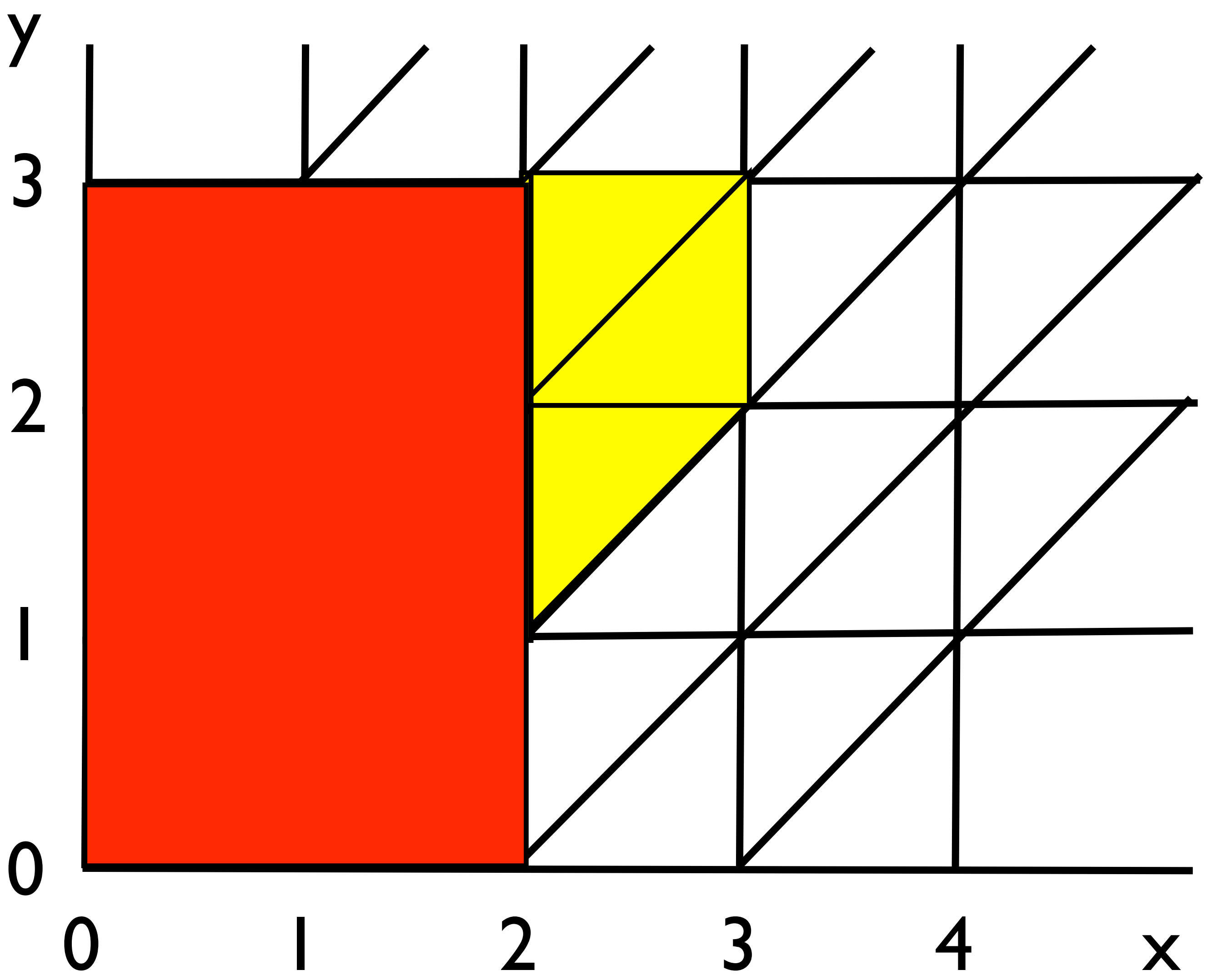
Reseting of y: $x \geq 1 \wedge x \leq 3 \wedge y=0$

Zones



Conjunction with the guard: $x \leq 2 \wedge y \leq 3$

Zones



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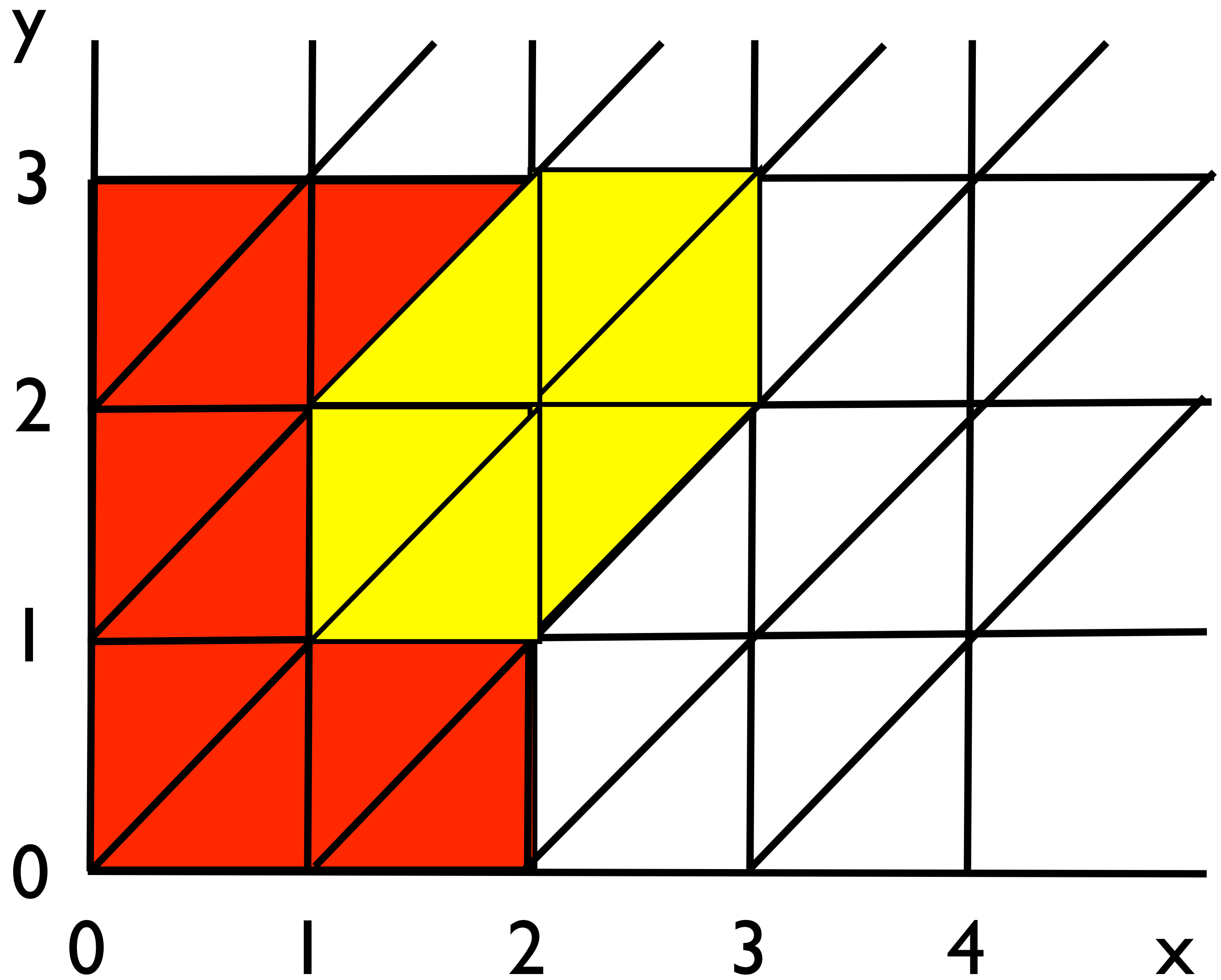
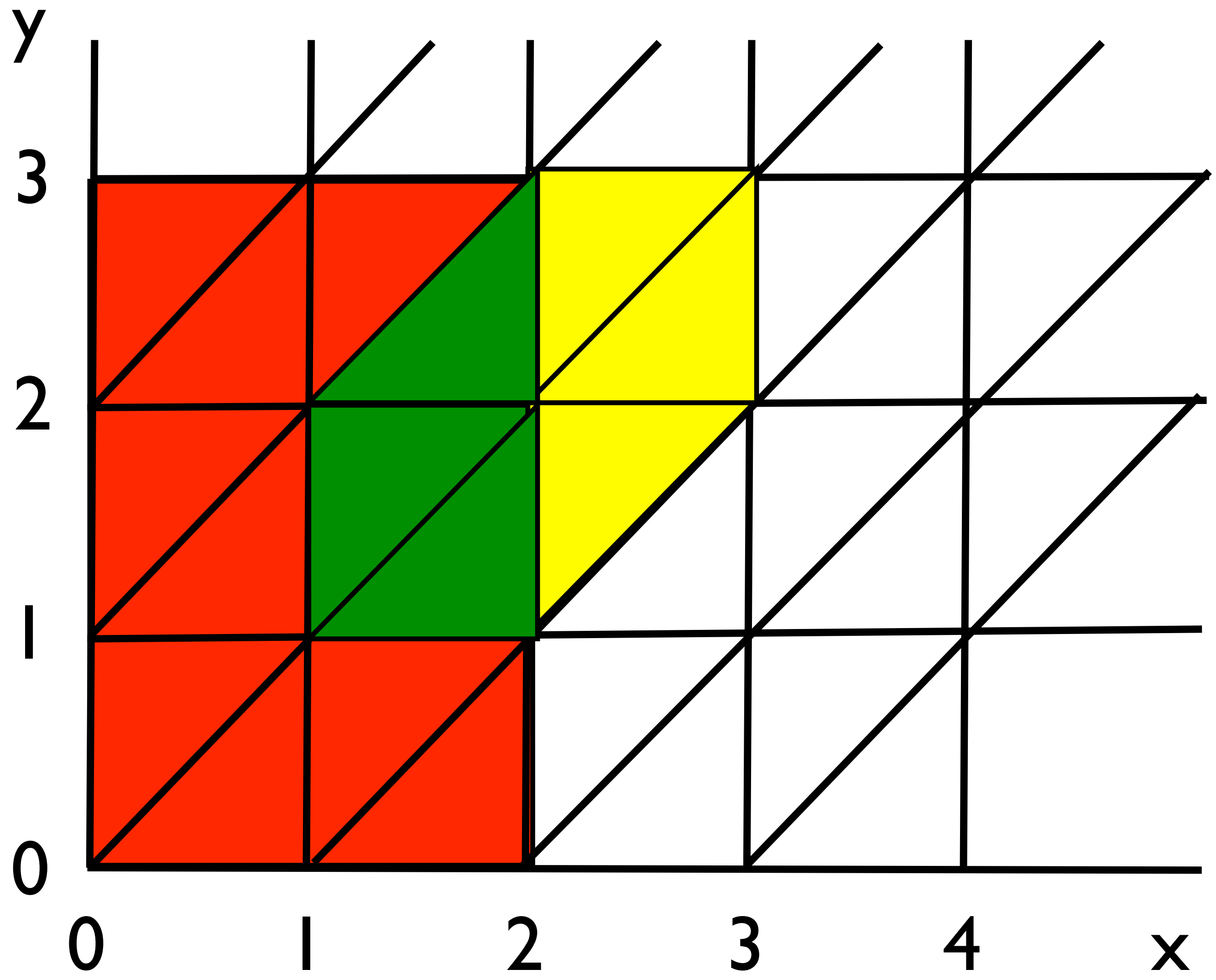
Zones



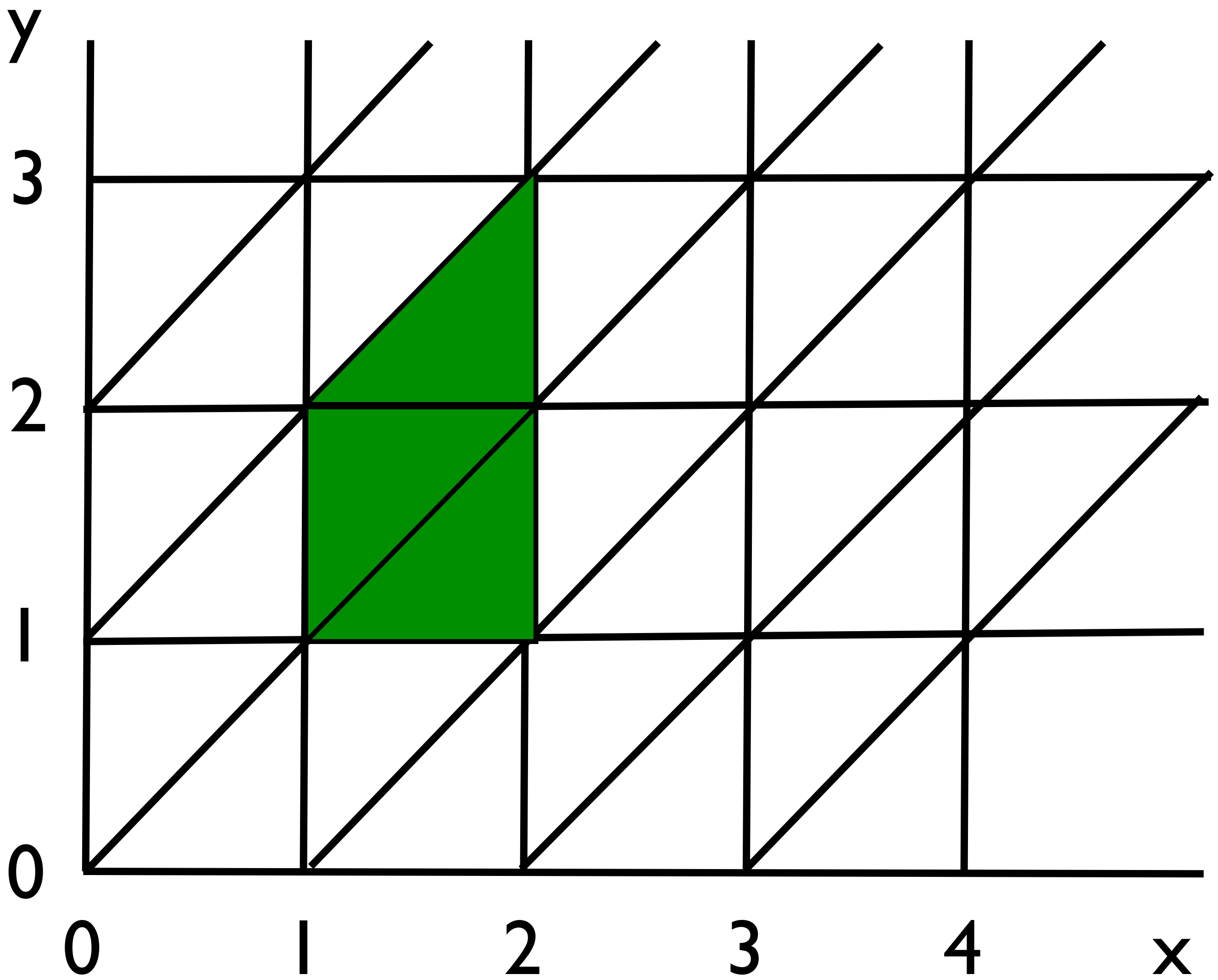
Conjunction with the guard: $x \leq 2 \wedge y \leq 3$

Zones



Conjunction with the guard: $x \leq 2 \wedge y \leq 3$

Zones



$x \geq 1 \wedge x \leq 2 \wedge y \geq 1 \wedge y \leq 3 \wedge x\text{-}y \geq \text{-}1$

Decidability results for TA

Decidability results

- As a direct consequence of our previous developments, we have that:
- The **reachability verification problem** for TA w.r.t. a set **Goal** which is defined as a union of regions is **decidable**.
- The **safety verification problem** for TA w.r.t. a set **Safe** which is defined as a union of regions is **decidable**.
- The **Büchi verification problem** for TA w.r.t. a set **Goal** which is defined as a union of regions is **decidable**.

Decidability results

- As the reachability verification problem for TA is decidable then the **timed language emptiness problem** (finite word case) is decidable for TA.

- As the Büchi verification problem for TA is decidable then the **timed language emptiness problem** (infinite word case) is decidable for TA.

  Hint to establish the result: construct a set Goal that ensures non-zenoness and the Büchi acceptance condition of the TA. Show that this set is a finite union of regions. As an intermediary step you will need a generalized Büchi condition.

Undecidability results for TA

Undecidability results for TA

- The **timed universality problem**, i.e. does a TA accepts all possible timed words on a alphabet, is **undecidable**.
- The **language inclusion problem** between timed automata is **undecidable**. (direct consequence of the previous undecidability result).