

Introduction to Control

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Outline

1. Introduction
2. Basic concepts
3. Modeling and design
4. Empirical PID control
5. Digital control

Automatic control

The silent technology:

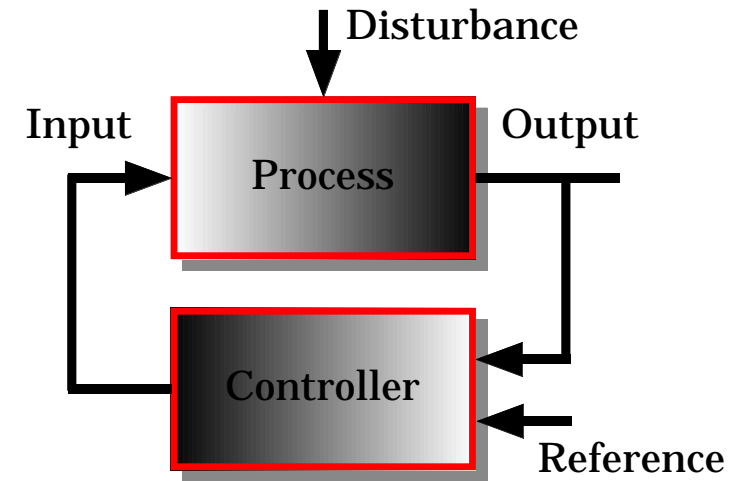
- Widely used
- Very successful
- Seldom talked about, except when disaster strikes!

Automatic control

Use of **models** and **feedback**

Activities:

- Modeling
- Analysis and simulation
- Control design
- Implementation



Applications

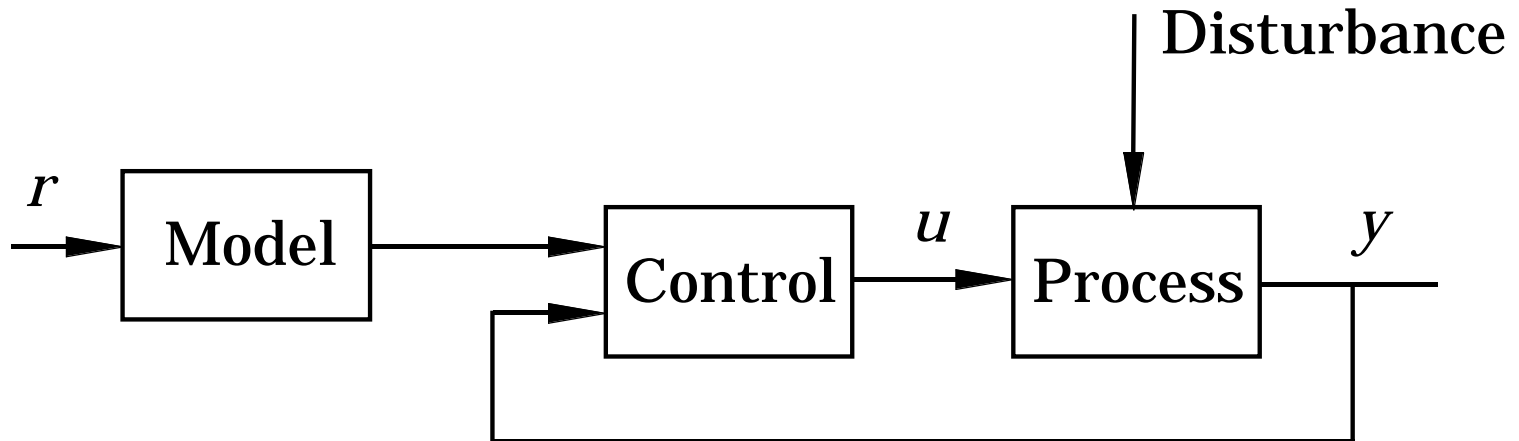
- Automotive systems
- Robotics
- Biotechnology
- Power systems
- Process control
- Communications
- Consumer electronics
- ...



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Basic setting



Must handle two tasks:

- Follow reference signals, r
- Compensate for disturbances

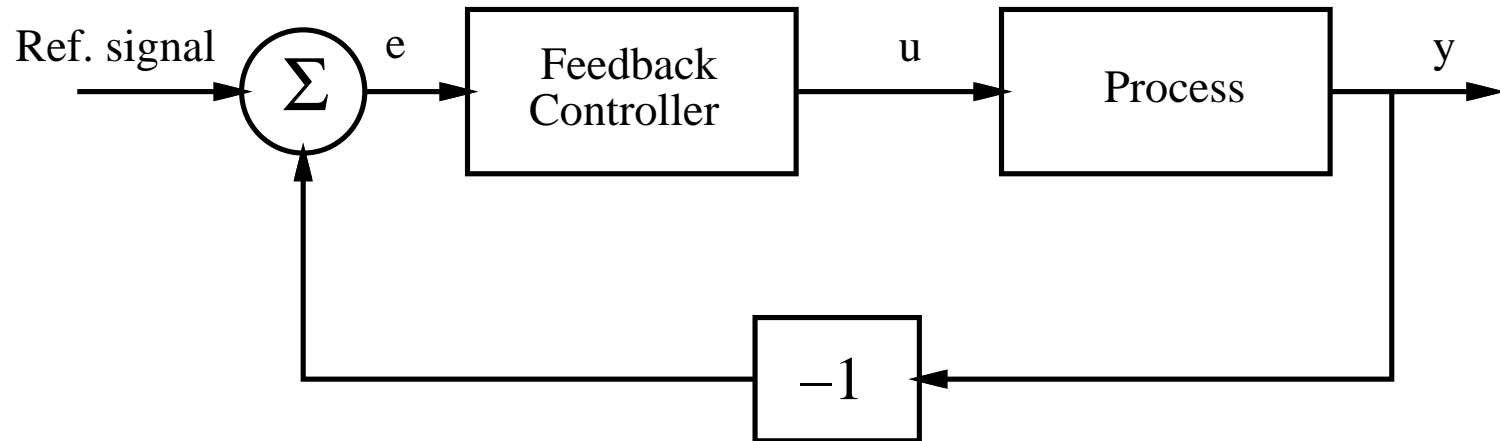
How to

- do several things with the control signal u

The feedback principle

A very powerful idea, that often leads to revolutionary changes in the way systems are designed.

The primary paradigm in automatic control.

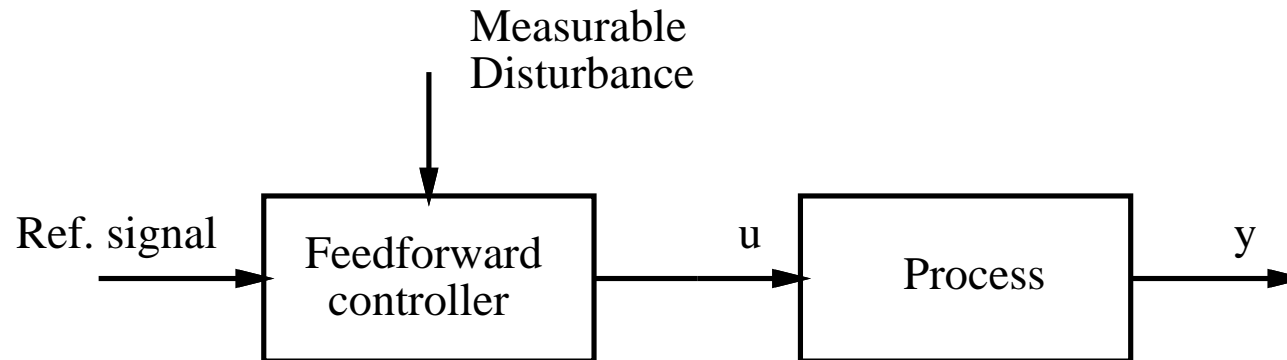


- Base corrective action on an error that has occurred
- Closed loop

Properties of feedback

- + Reduces influence of disturbances
- + Reduces effect of process variations
- + Does not require exact models
- Feeds sensor noise into the system
- May lead to instability, e.g.:
 - if the controller has too high gain
 - if the feedback loop contains too large time delays
 - * from the process
 - * from the controller implementation

The feedforward principle



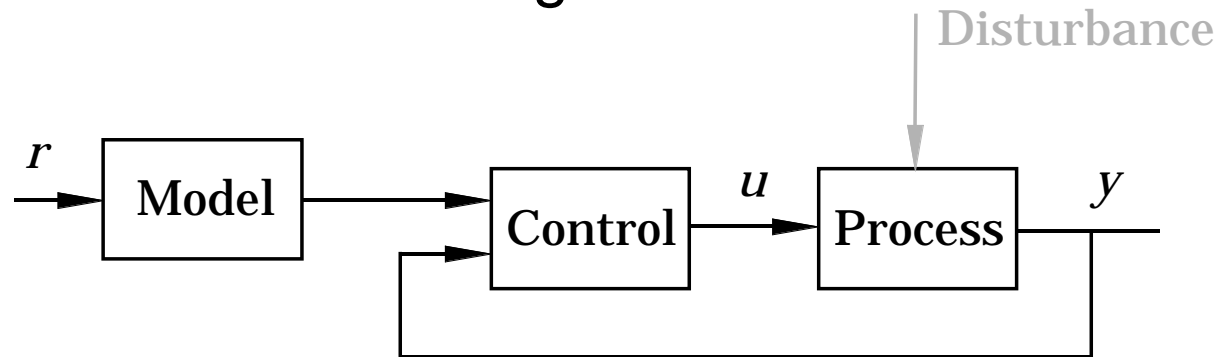
- Take corrective action before an error has occurred
- Measure the disturbance and compensate for it
- Use the fact that the reference signal is known and adjust the control signal to the reference signal
- Open loop

Properties of feedforward

- + Reduces effect of disturbances that cannot be reduced by feedback
- + Measurable signals that are related to disturbances
- + Allows faster set-point changes, without introducing control errors
- Requires good models
- Requires stable systems

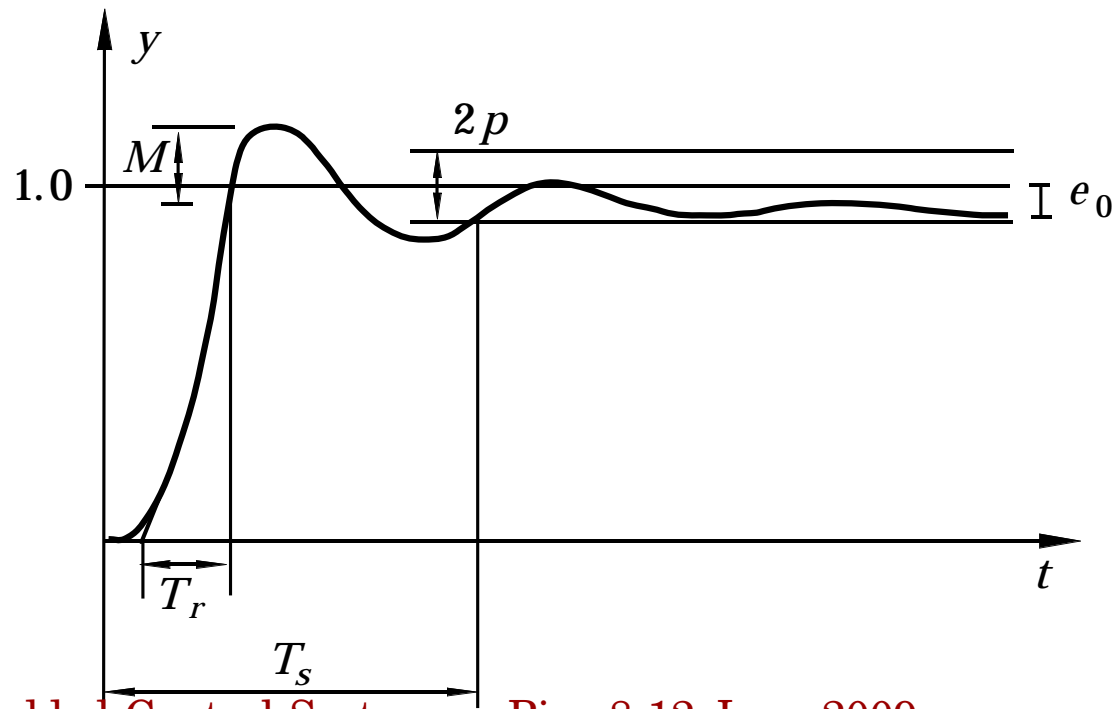
The servo problem

Focus on reference value changes:



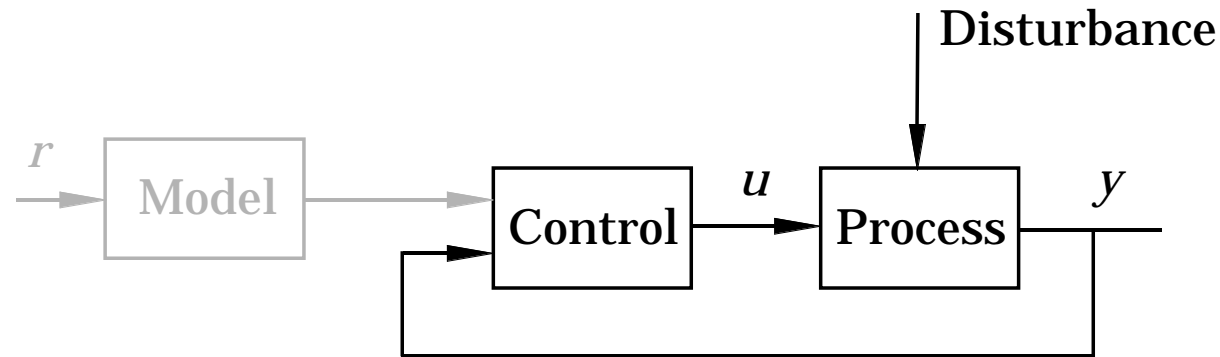
Typical design criteria:

- Rise time, T_r
- Overshoot, M
- Settling time, T_s
- Steady-state error, e_0
- ...



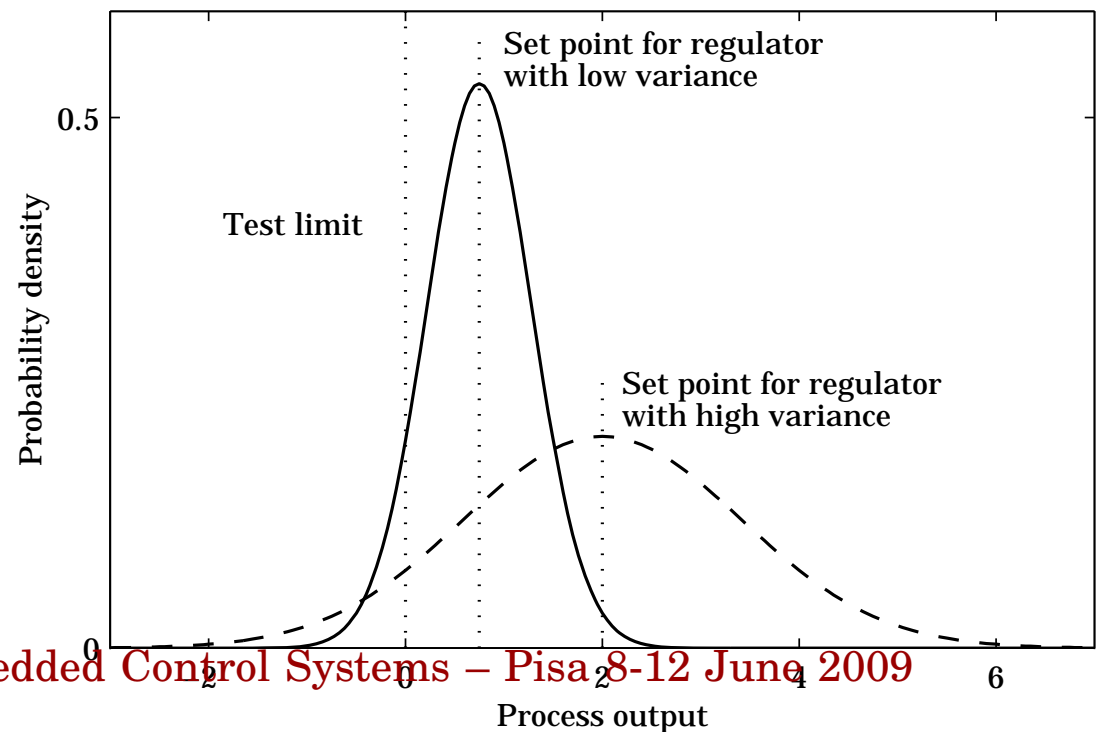
The regulator problem

Focus on process disturbances:

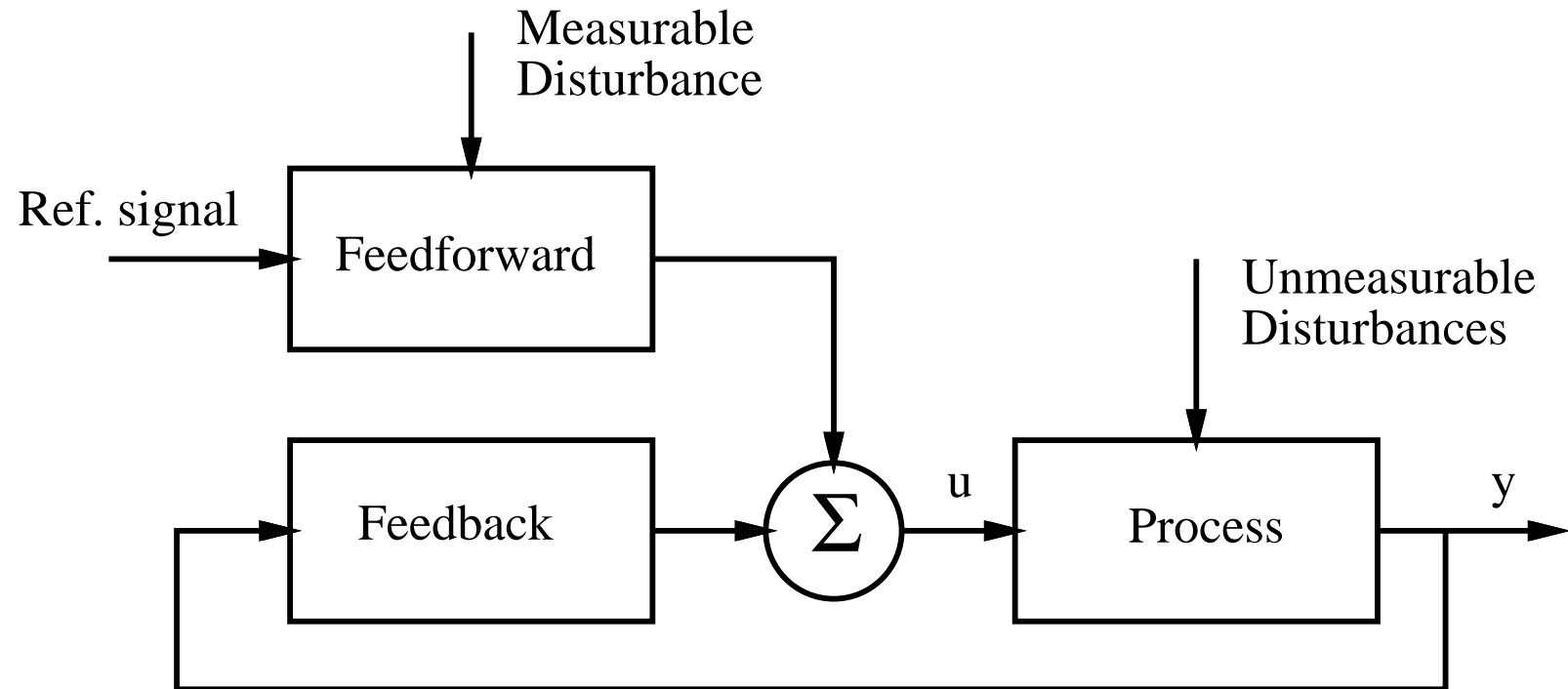


Typical design criteria:

- Output variance
- Control signal variance



Putting it all together



Combination of **feedback** and **feedforward**

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Dynamical systems



Static system:

$$y(t) = f(u(t))$$

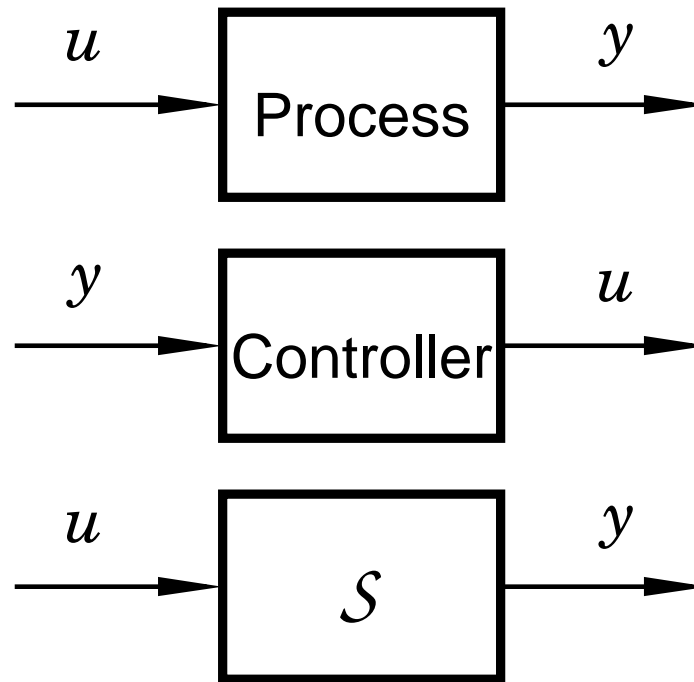
(The output at time t only depends on the input at time t .)

Dynamical system:

$$y(t) = f(x(0), u_{[0,t]})$$

(The output at time t depends on the initial state $x(0)$ and the input from time 0 to t .)

Modeling

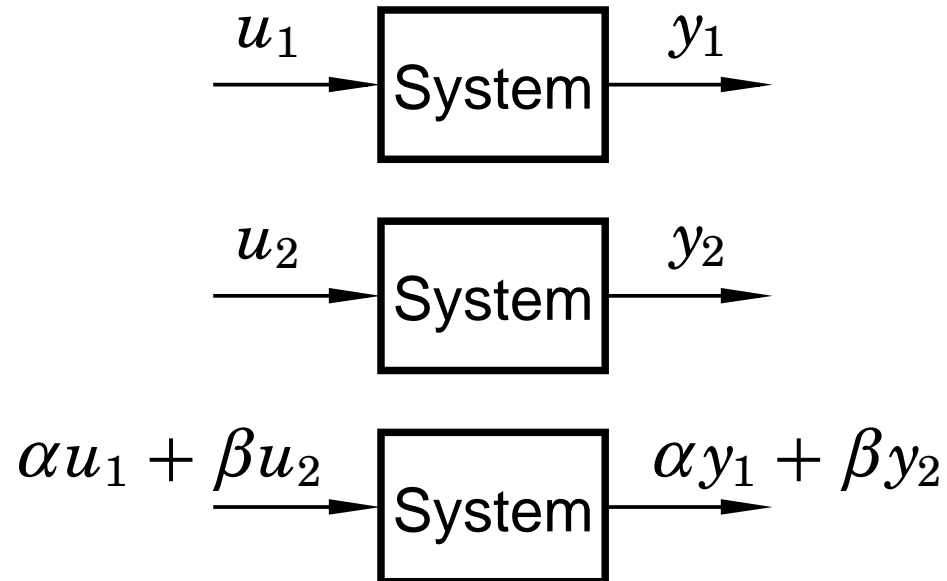


- View all subsystems as “boxes” with inputs and outputs
- Linear, time-invariant (LTI) dynamical systems
- Continuous or discrete time

Linear systems

We will mainly deal with linear, time-invariant (LTI) systems

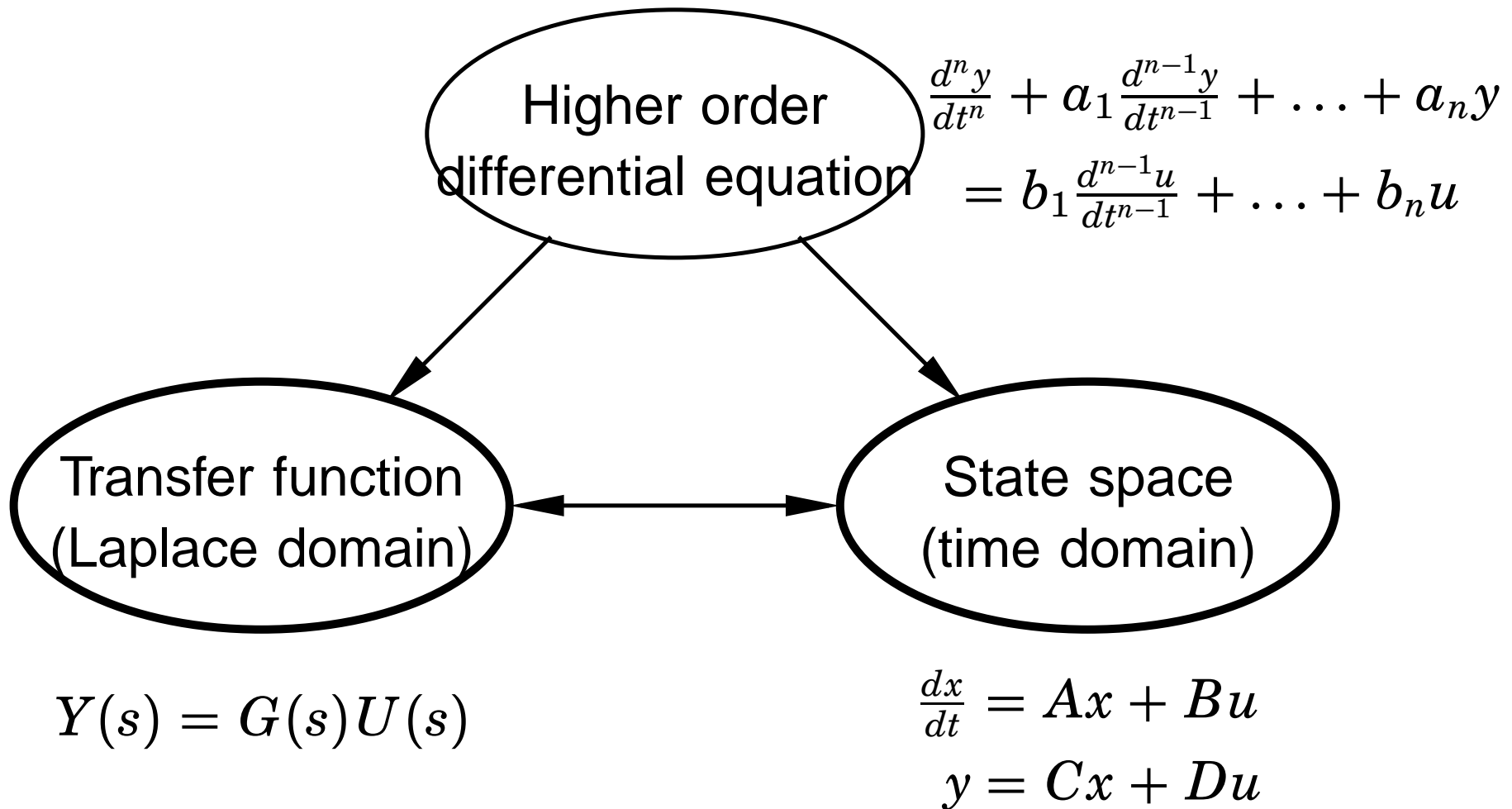
For linear systems, the principle of superposition holds:



Nonlinear systems

- Almost all real systems are nonlinear
 - limited input and output signals
 - nonlinear process geometry
 - friction, turbulence, ...
- Can be linearized around an operating point
- If there is feedback, a simple linear model is often enough
- But, always remember the limitations of the model!

Continuous-time systems



Standard system forms

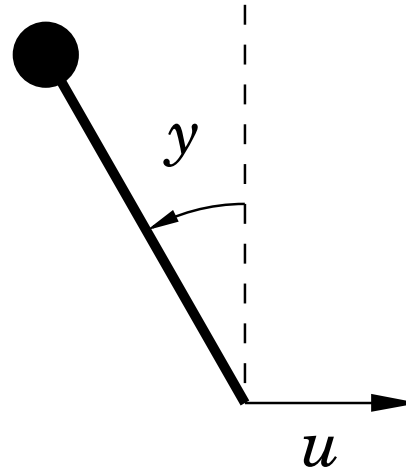
- State space form
 - A number of first-order differential equations
 - Describes what happens “inside” the system and how inputs and output are connected to this
 - Numerically superior
 - The heritage of mechanics
- Transfer function form
 - The transform of a higher-order linear differential equation
 - Describes the relationship between the input and the output
 - The system is a “black box”
 - Compact notation, convenient for hand calculations



The heritage of electrical engineering

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Example: Inverted pendulum



Nonlinear differential equation from physical modeling:

$$\frac{d^2 y}{dt^2} = \omega_0^2 \sin y + k u \cos y$$

Linearized model around $y^0 = 0$ ($\sin y \approx y$, $\cos y \approx 1$):

$$\frac{d^2 y}{dt^2} = \omega_0^2 y + k u$$

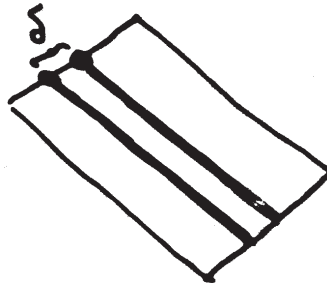
Inverted pendulum in state space form

Introduce state variables

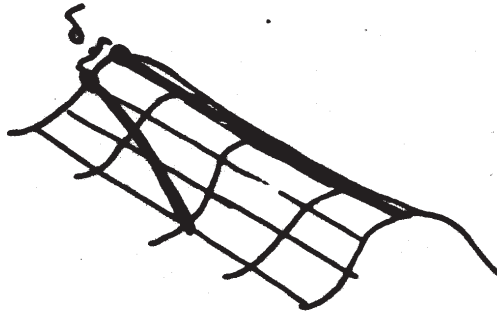
- $x_1 = y$ (pendulum angle)
- $x_2 = \frac{dy}{dt}$ (pendulum angular velocity)

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

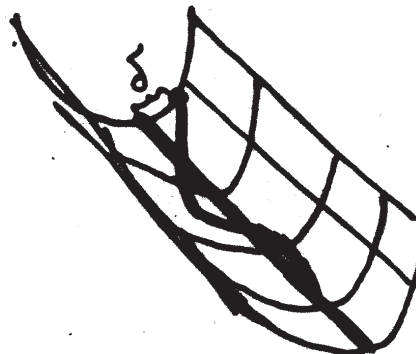
Stability concepts



Stable



Unstable



Asymptotically stable

Stability definitions

Assume

$$\dot{x} = Ax, \quad x(0) = x_0$$

The system is **stable** if $x(t)$ is limited for all x_0 .

The system is **asymptotically stable** if $x(t) \rightarrow 0$ for all x_0 .

The system is **unstable** if $x(t)$ is unlimited for some x_0 .

Stability criteria

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = x_0 e^{At}$$

The behavior of the solution depends on the eigenvalues of A

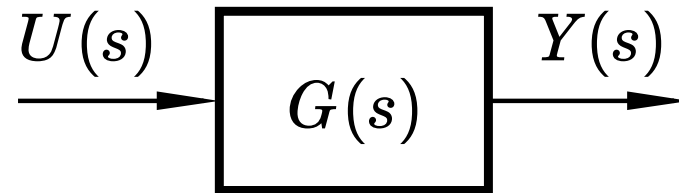
All eigenvalues have negative real part: \Leftrightarrow As. stab.

Some eigenvalue has positive real part: \Rightarrow Unstable

No eigenvalues with positive real part and no multiple eigenvalues on the imaginary axis: \Leftrightarrow Stable

Transfer function form

Study the system in the (complex) frequency domain:



$U(s)$ – Laplace transform of $u(t)$

$Y(s)$ – Laplace transform of $y(t)$

$G(s)$ – transfer function

$$Y(s) = G(s)U(s)$$

(if the initial state is assumed to be zero)

Some signals and their Laplace transforms

Definition: $\mathcal{L}f = F(s) = \int_0^{\infty} e^{-st} f(t) dt$

Derivative: $\mathcal{L}\left(\frac{df}{dt}\right) = sF(s)$

Integral: $\mathcal{L}\left(\int f dt\right) = \frac{1}{s}F(s)$

Dirac impulse: $\mathcal{L}\delta = 1$

Step function: $\mathcal{L}\theta = \frac{1}{s}$

Ramp function: $\mathcal{L}(t\theta) = \frac{1}{s^2}$

Exponential function: $\mathcal{L}(e^{at}\theta) = \frac{1}{s-a}$

From transfer function to state space form

$$\begin{cases} \dot{x} = Ax + Bu & x(0) = 0 \\ y = Cx + Du \end{cases}$$

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$Y(s) = [C(sI - A)^{-1}B + D] U(s)$$

$$G(s) = C(sI - A)^{-1}B + D = \frac{p(s)}{q(s)}$$

$q(s) = \det(sI - A)$ is called **characteristic polynomial**

Poles and zeros

Often,

$$G(s) = \frac{p(s)}{q(s)}$$

The roots of $p(s)$ are called zeros

The roots of $q(s)$ are called poles

Note that

$$\text{Poles of } G(s) \quad \Leftrightarrow \quad \text{Eigenvalues of } A$$

Inverted pendulum in transfer function form

Apply Laplace transform to differential equation:

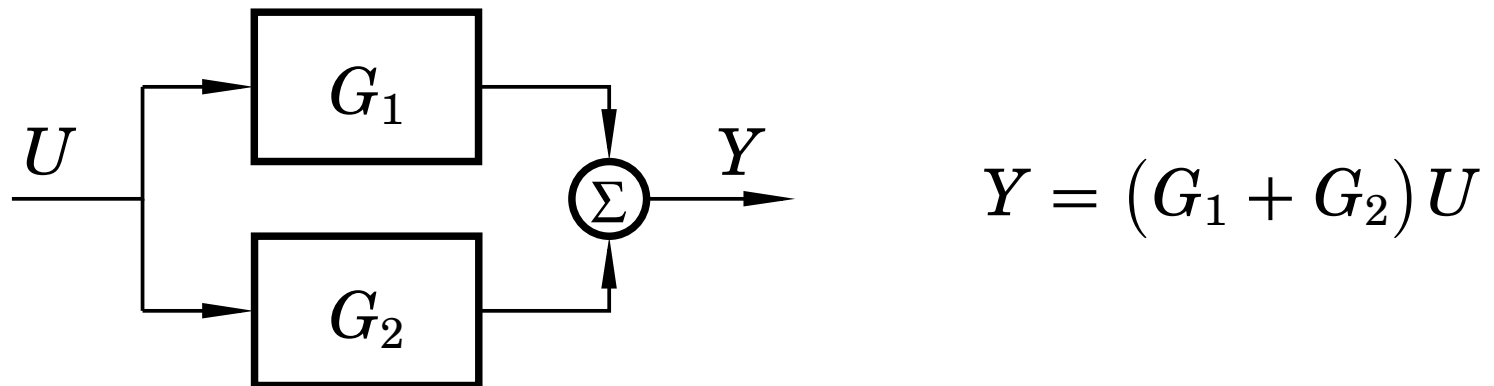
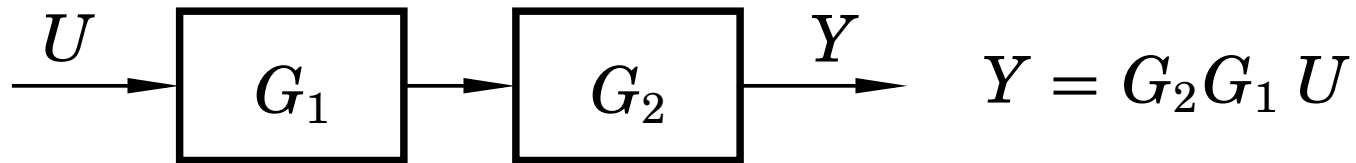
$$s^2 Y(s) = \omega_0^2 Y(s) + k U(s)$$

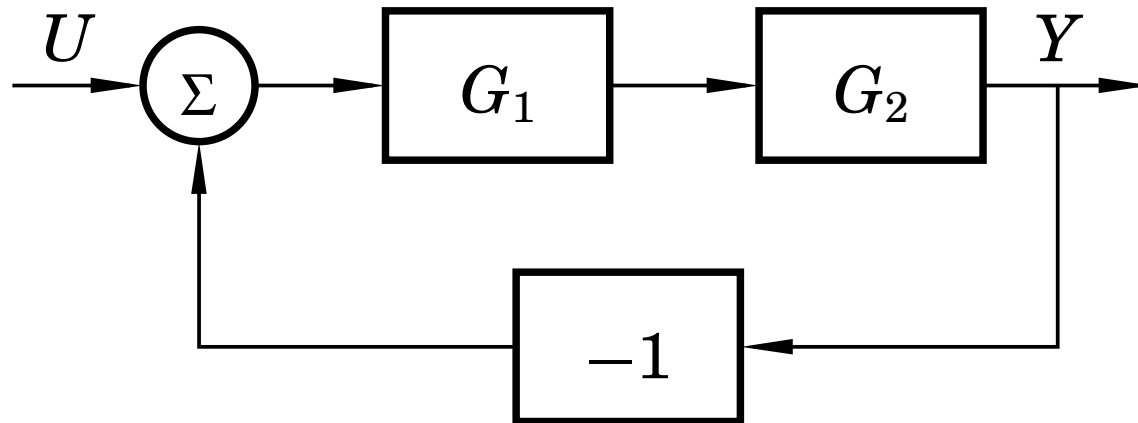
$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^2 - \omega_0^2}$$

Or, from state space to transfer function:

$$G(s) = C(sI - A)^{-1}B = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ -\omega_0^2 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ k \end{pmatrix} = \frac{k}{s^2 - \omega_0^2}$$

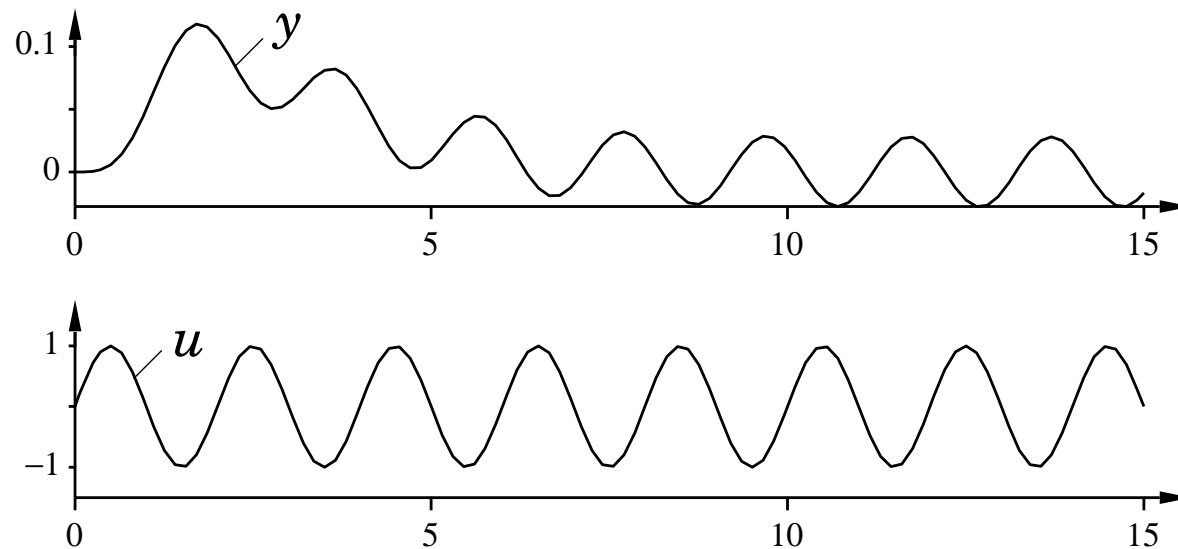
Block diagrams





$$Y = G_2 G_1 (U - Y)$$
$$Y(1 + G_2 G_1) = G_2 G_1 U$$
$$Y = \frac{G_2 G_1}{1 + G_2 G_1} U$$

Frequency response



Given a stable system $G(s)$, the input $u(t) = \sin \omega t$ will, after a transient, give the output

$$y(t) = |G(i\omega)| \sin(\omega t + \arg G(i\omega))$$

The steady-state output is also sinusoidal

Bode diagram

Draw

- $|G(i\omega)|$ as a function of ω (in log-log scale)
 - Amplitude/magnitude/gain diagram
- $\arg G(i\omega)$ as a function of ω (in log-lin scale)
 - Phase/angle diagram

Example: low-pass filter

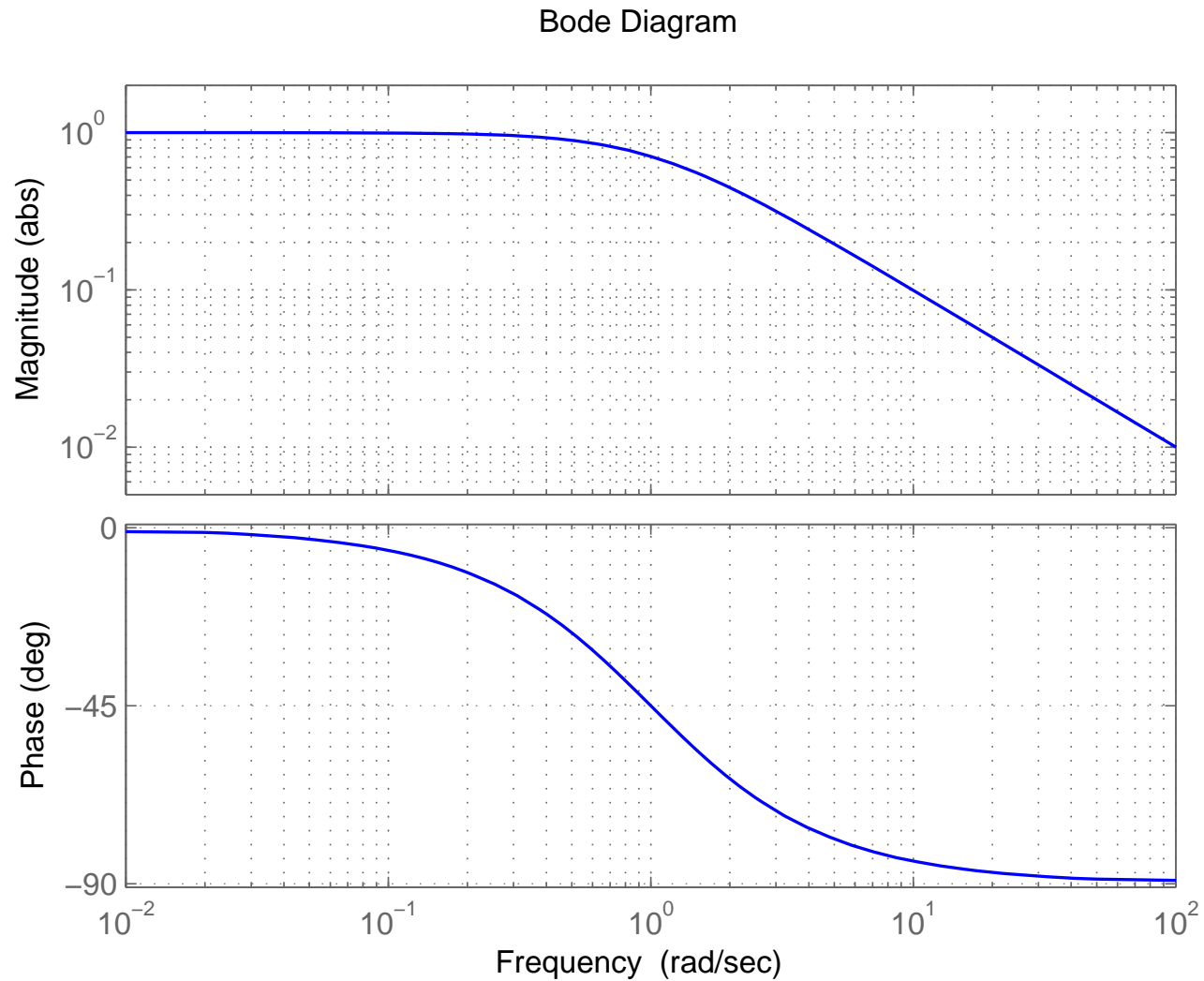
$$\frac{dy(t)}{dt} + y(t) = u(t) \quad \Leftrightarrow \quad G(s) = \frac{1}{s + 1}$$

$$G(i\omega) = \frac{1}{i\omega + 1}$$

$$|G(i\omega)| = \frac{1}{\sqrt{\omega^2 + 1}}$$

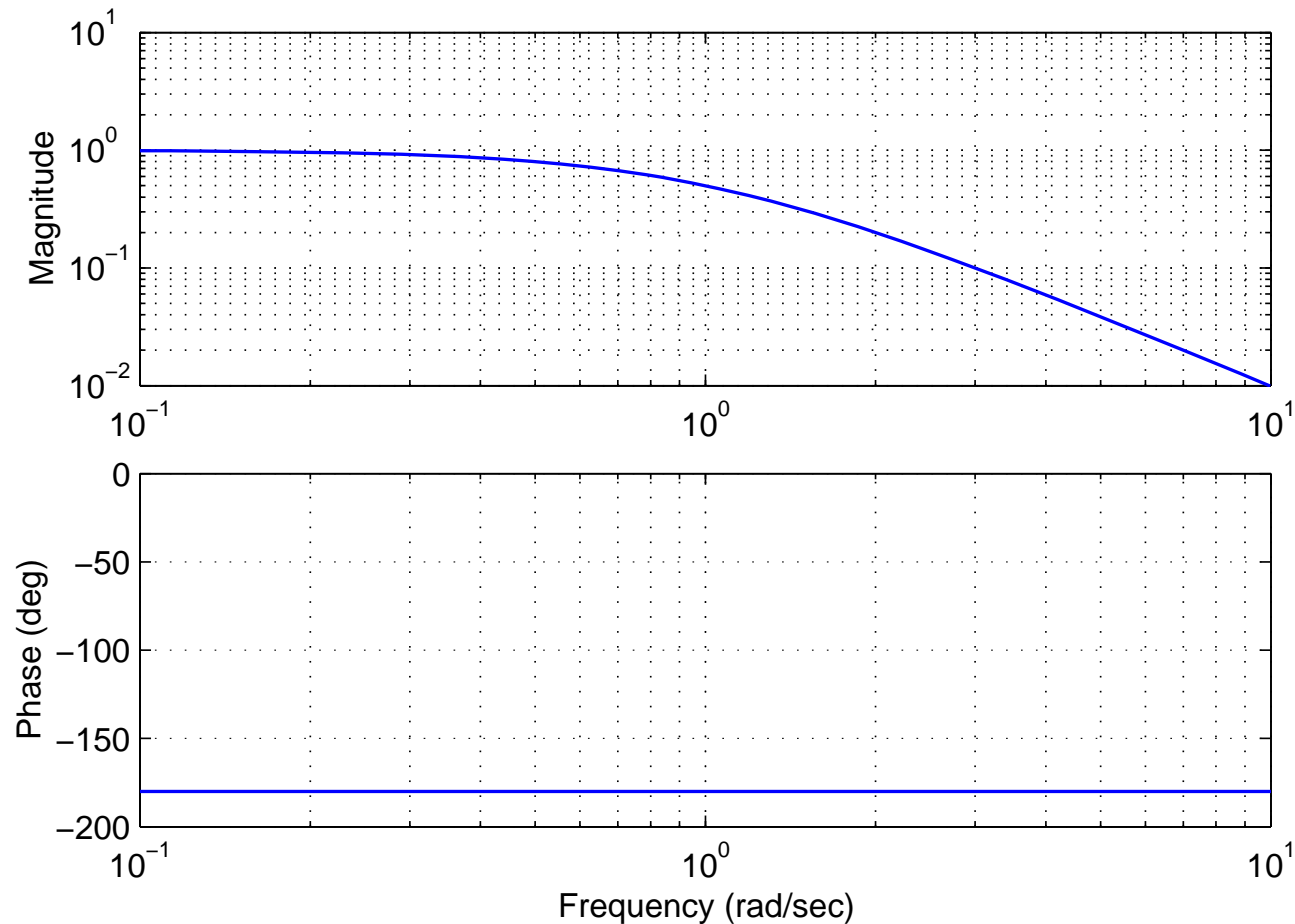
$$\arg G(i\omega) = -\arctan \omega$$

Example: low-pass filter

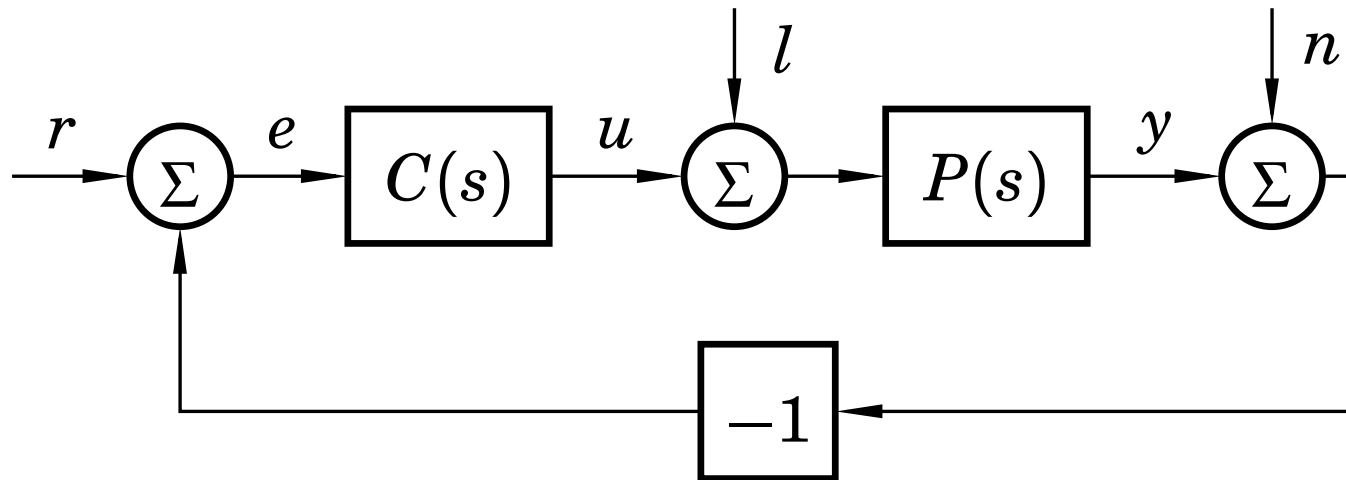


Frequency response for inverted pendulum

Plot $|G(i\omega)|$ and $\arg G(i\omega)$ for $\omega \in [0, \infty]$



Model-based design



Given $P(s)$, determine $C(s)$ such that the specifications on the closed-loop system are met. Common approaches:

- Frequency domain design (loop shaping)
- Pole placement design
 - transfer function domain
 - state space domain

• Optimization-based methods (H_∞ , LQG, ...)

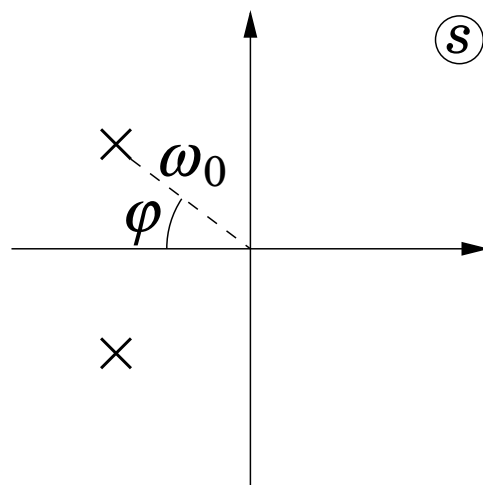
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Pole placement – transfer function domain

- Determine the required form of $C(s) = \frac{b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$
- Calculate the closed loop system:

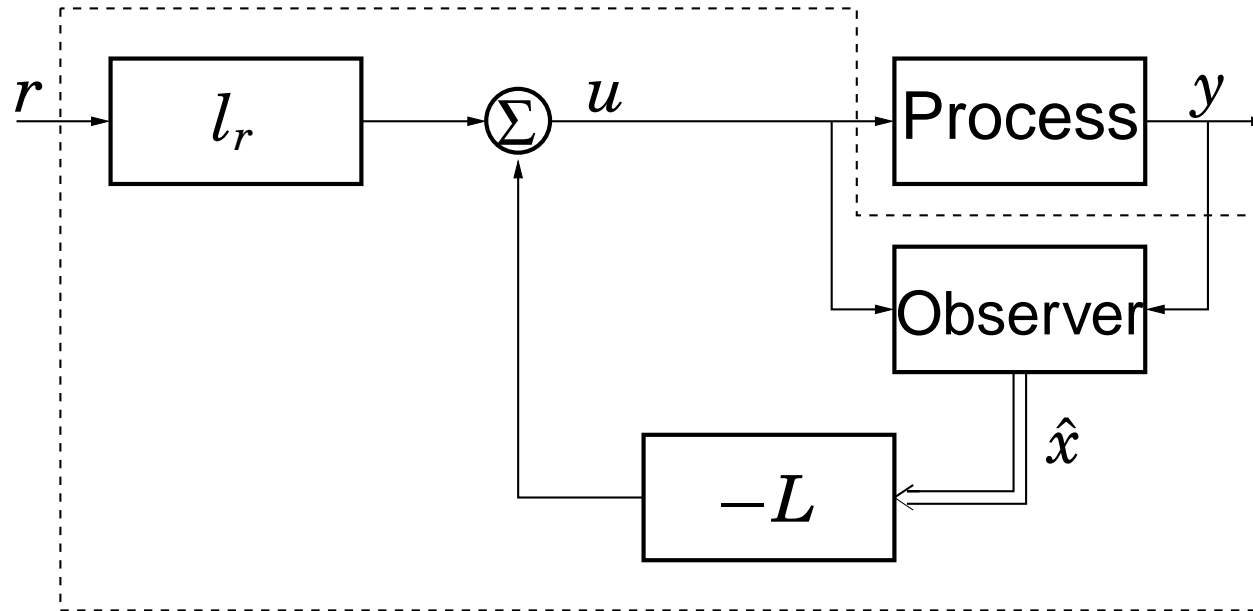
$$G_{cl}(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$

- Choose the coefficients of $C(s)$ such that you get the desired closed-loop poles



- Large $\omega_0 \Leftrightarrow$ fast system
- Large $\varphi \Leftrightarrow$ poorly damped system

Pole placement – state space domain



State feedback from an observer:

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

$$u = -L\hat{x} + l_r r$$

Choose gain vectors L and K to give desired closed-loop poles

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PID control

- The oldest controller type
- The most widely used
 - Pulp & Paper 86%
 - Steel 93%
 - Oil refineries 93%
- Much to learn!!

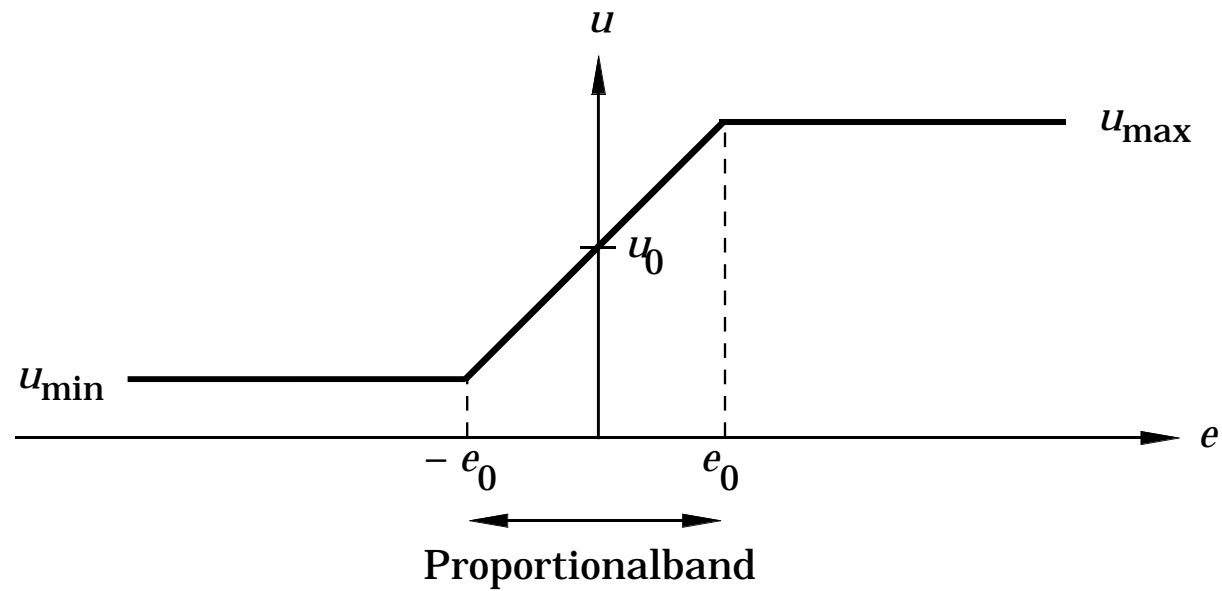
The textbook algorithm

$$u(t) = K \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right)$$

$$U(s) = K \left(E(s) + \frac{1}{sT_i} E(s) + T_d s E(s) \right)$$

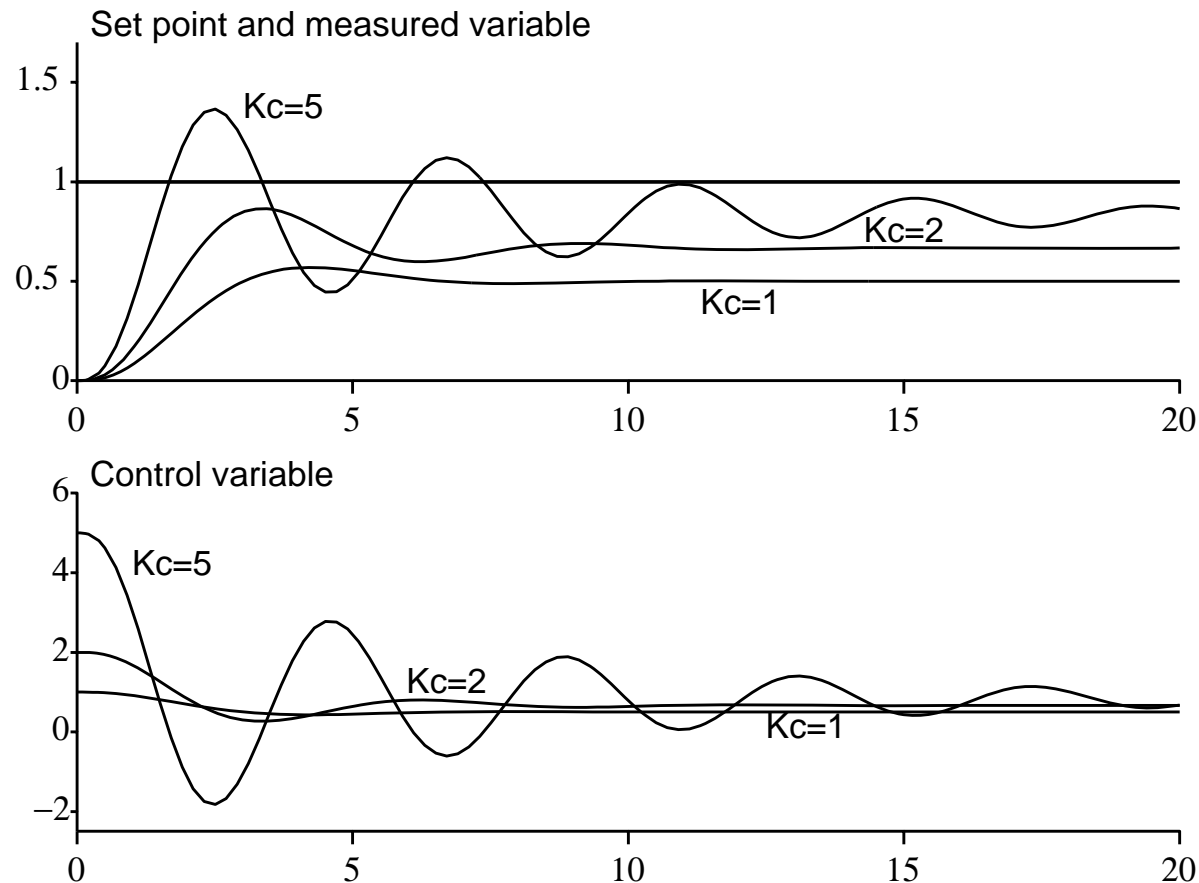
$$= P + I + D$$

Proportional term



$$u = \begin{cases} u_{\max} & e > e_0 \\ K e + u_0 & -e_0 < e < e_0 \\ u_{\min} & e < -e_0 \end{cases}$$

Properties of P-control



- stationary error
- increased K means faster speed, increased noise sensitivity, worse stability



Errors with P-control

Control signal:

$$u = K e + u_0$$

Error:

$$e = \frac{u - u_0}{K}$$

Error removed if:

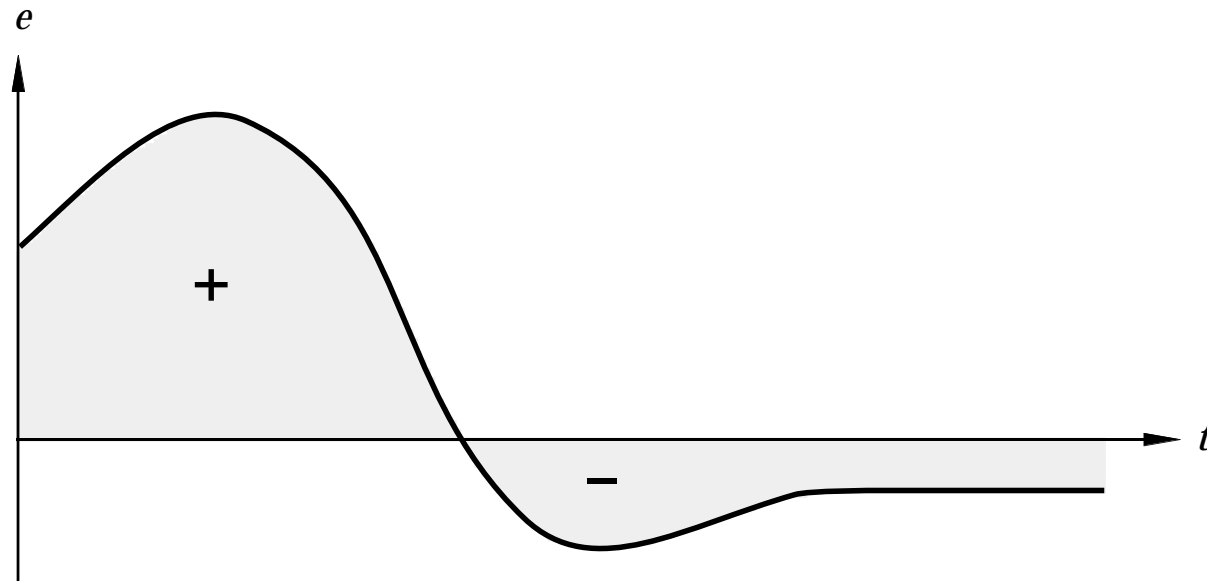
- K equals infinity
- $u_0 = u$

Solution: Automatic way to obtain u_0

Integral term

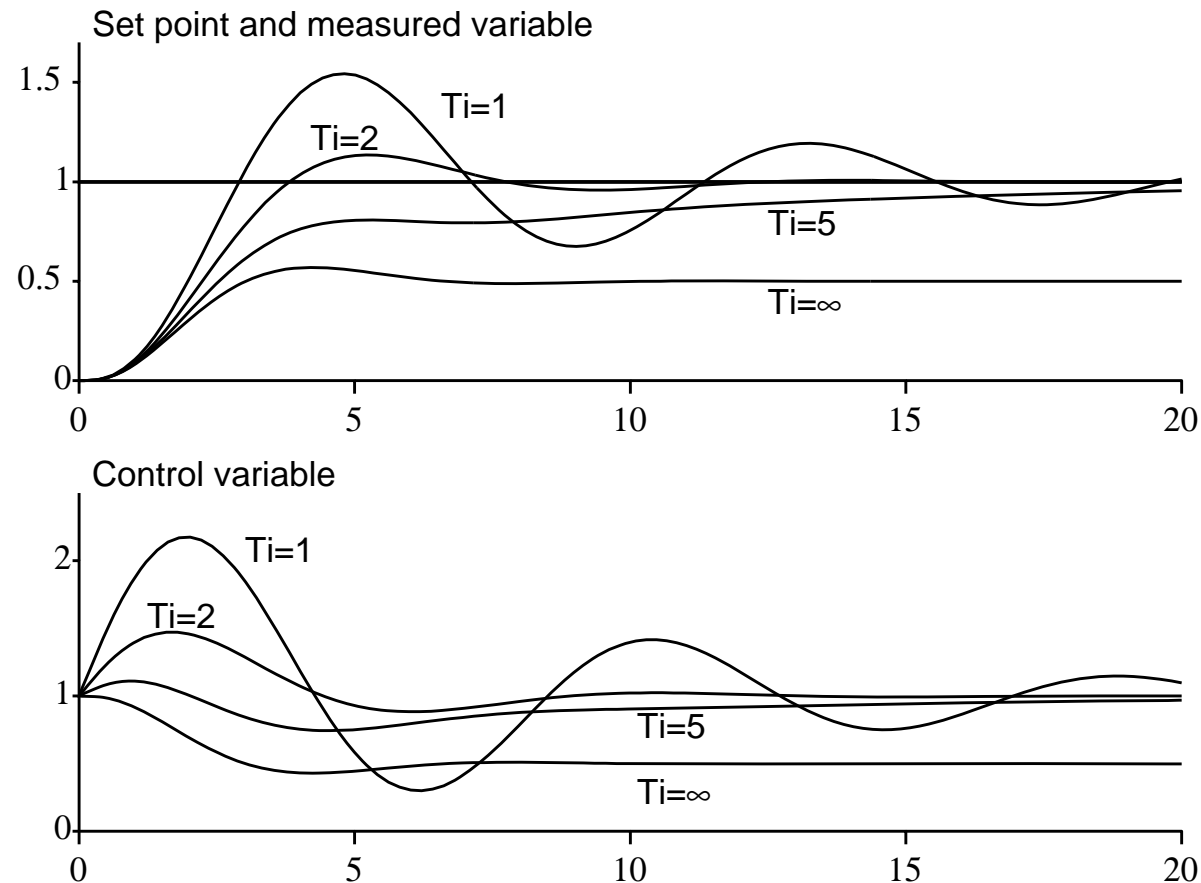
$$u = Ke + u_0$$

$$u = K \left(e + \frac{1}{T_i} \int e(t) dt \right) \quad (\text{PI})$$



Stationary error present $\rightarrow \int e dt$ increases $\rightarrow u$ increases $\rightarrow y$ increases \rightarrow the error is not stationary

Properties of PI-control

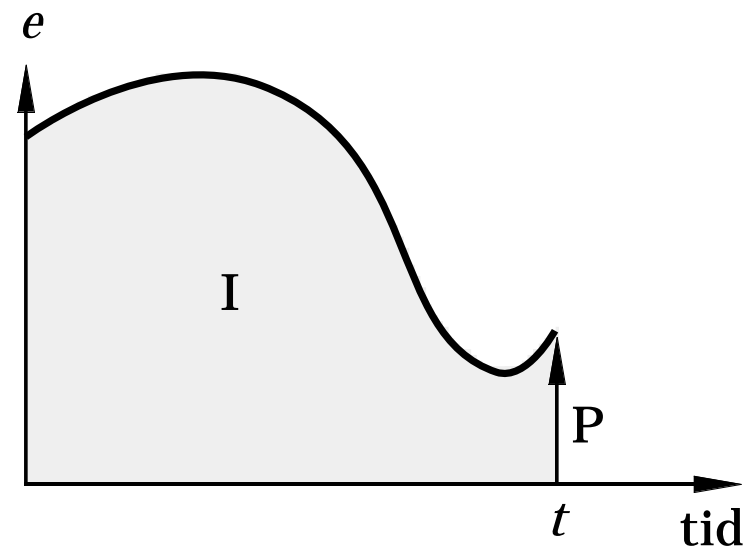
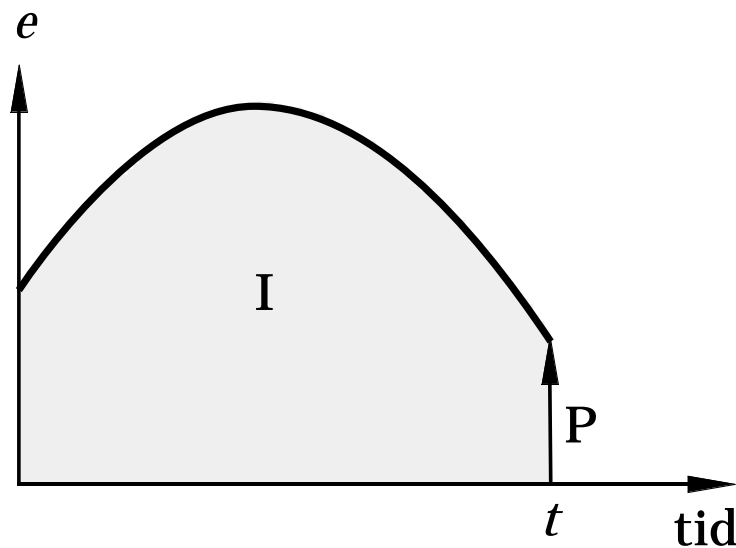


- removes stationary error
- smaller T_i implies worse stability, faster steady-state error

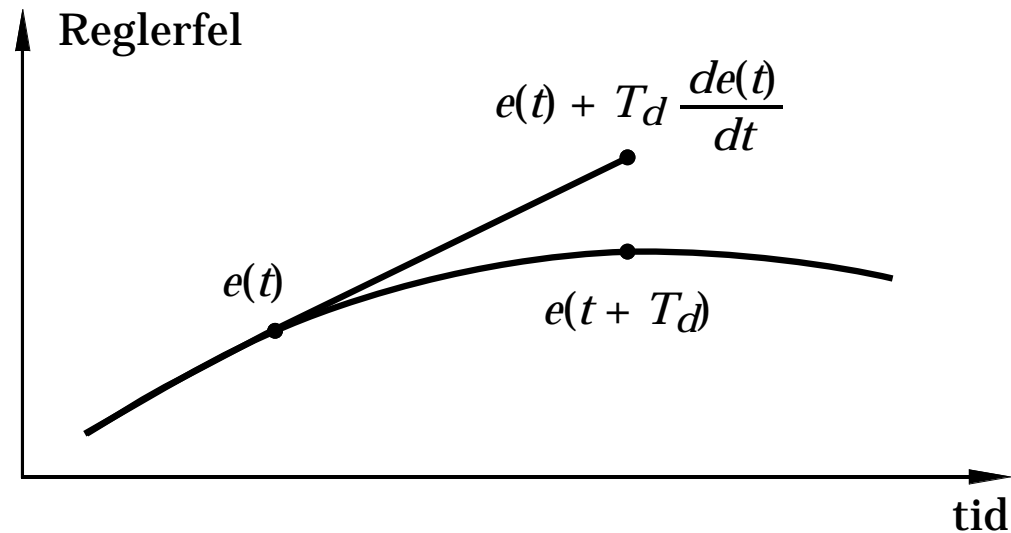
Prediction

A PI-controller contains no prediction

The same control signal is obtained for both these cases:



Derivative part



P:

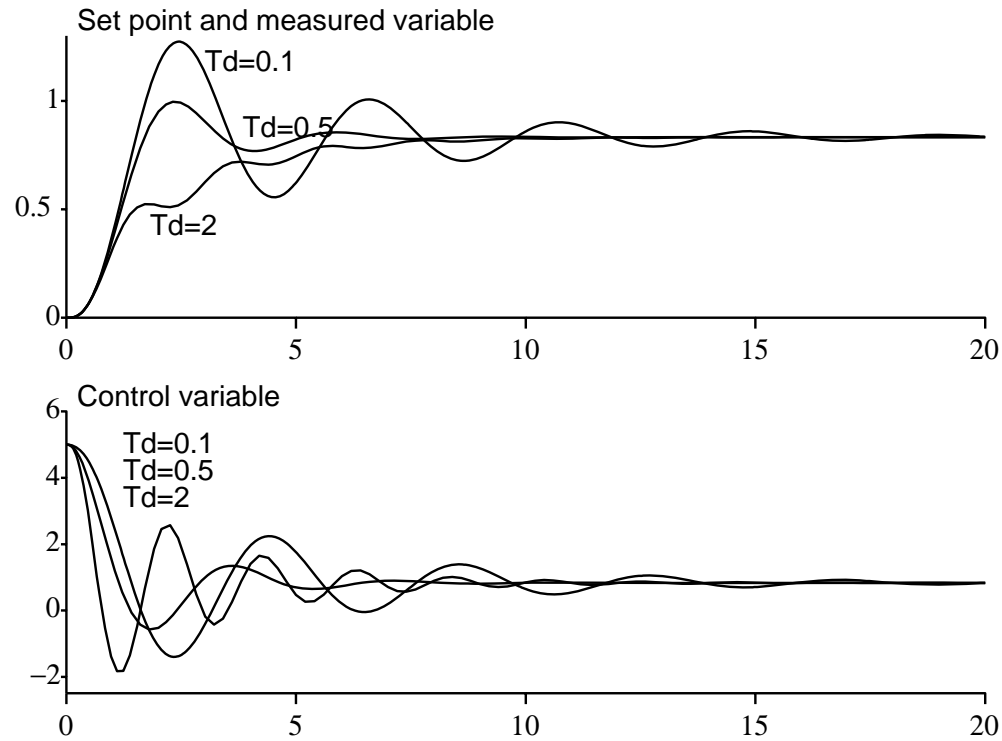
$$u(t) = K e(t)$$

PD:

$$u(t) = K \left(e(t) + T_d \frac{de(t)}{dt} \right) \approx K e(t + T_d)$$

T_d = Prediction horizon

Properties of PD-control



- T_d too small, no influence
- T_d too large, decreased performance

In industrial practice the D-term is often turned off.

Algorithm modifications

Modifications are needed to make the controller practically useful

- Limitations of derivative gain
- Derivative weighting
- Handle control signal limitations

Limitations of derivative gain

We do not want to apply derivation to high frequency measurement noise, therefore the following modification is used:

$$sT_d \approx \frac{sT_d}{1 + sT_d/N}$$

N = maximum derivative gain, often 10 – 20

Derivative weighting

The reference is often constant for long periods of time

Reference often changed in steps \rightarrow D-part becomes very large.

Derivative part applied on part of the reference or only on the measurement signal.

$$D(s) = \frac{sT_d}{1 + sT_d/N} (\gamma R(s) - Y(s))$$

Often, $\gamma = 0$

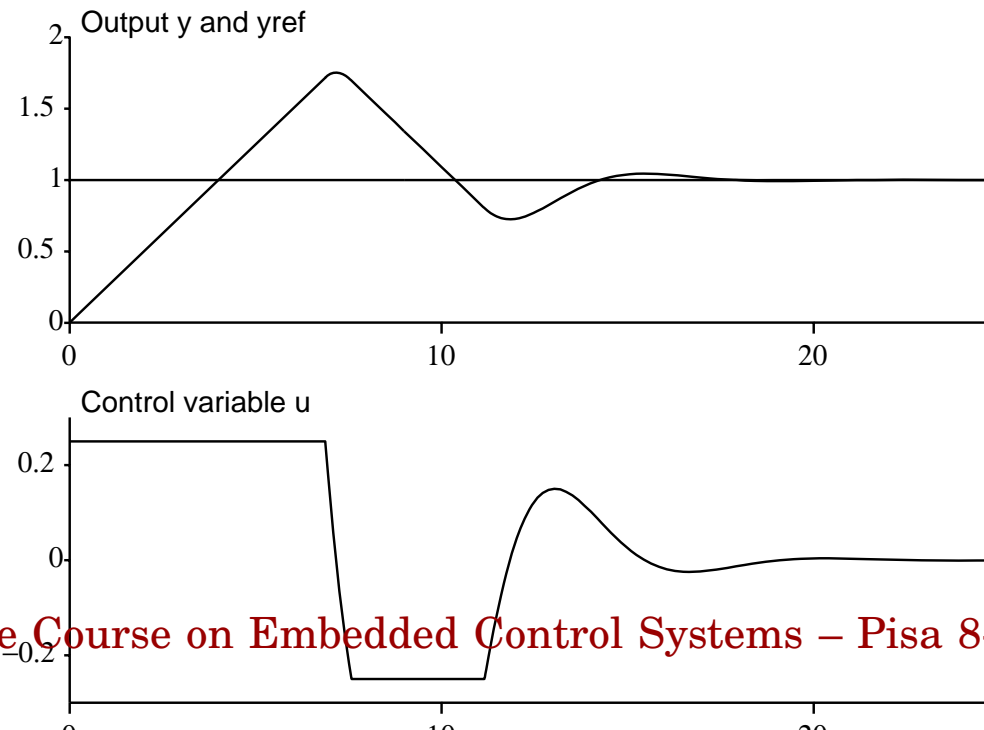
Control signal limitations

All actuators saturate.

Problems for controllers with integration.

When the control signal saturates the integral part will continue to grow – integrator windup.

When the control signal saturates the integral part will integrate up to a very large value. This may cause large overshoots.



Anti-windup

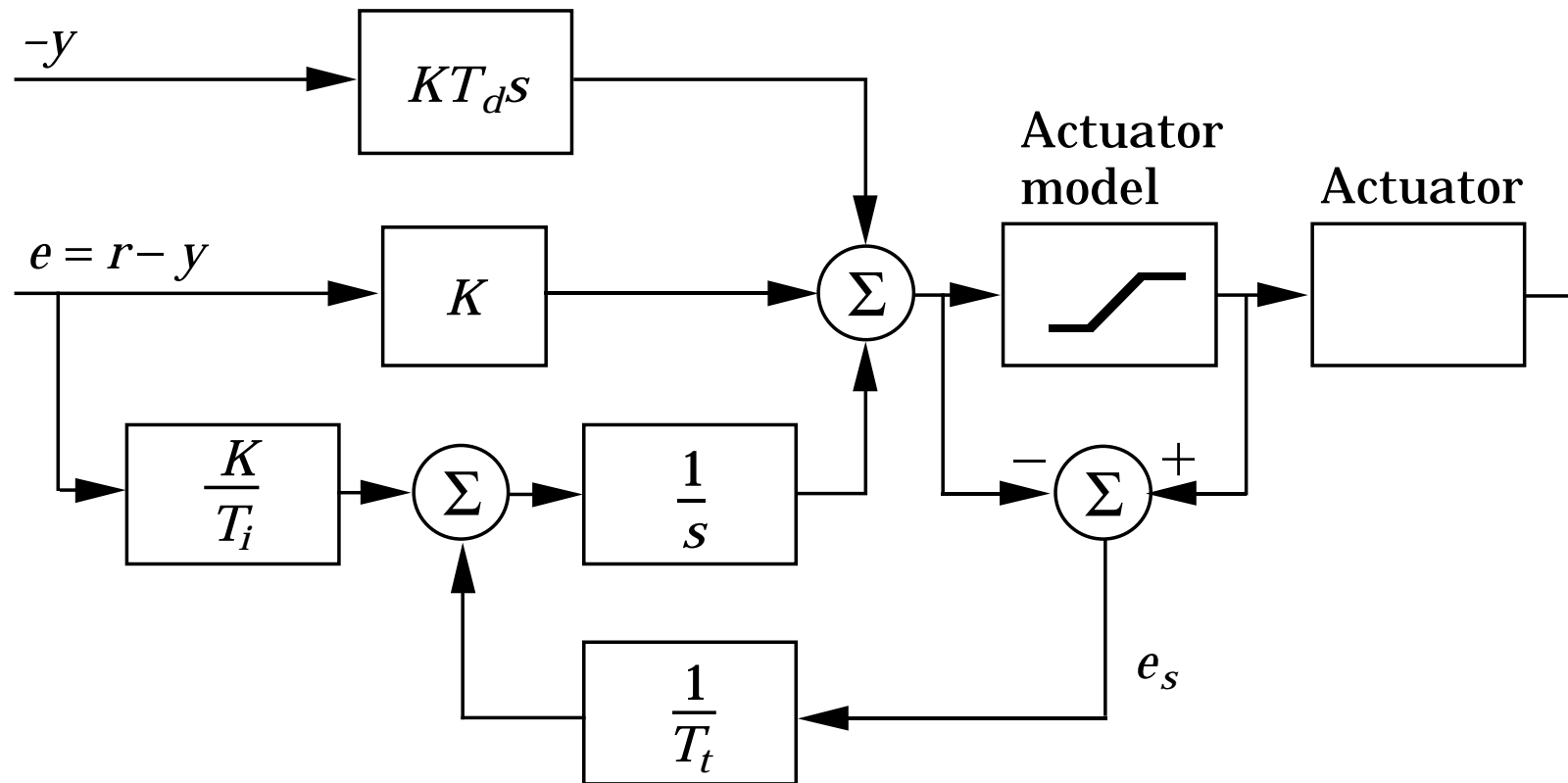
Several solutions exist:

- limit the reference variations (saturation never reached)
- conditional integration (integration is switched off when the control is far from the steady-state)
- tracking (back-calculation)

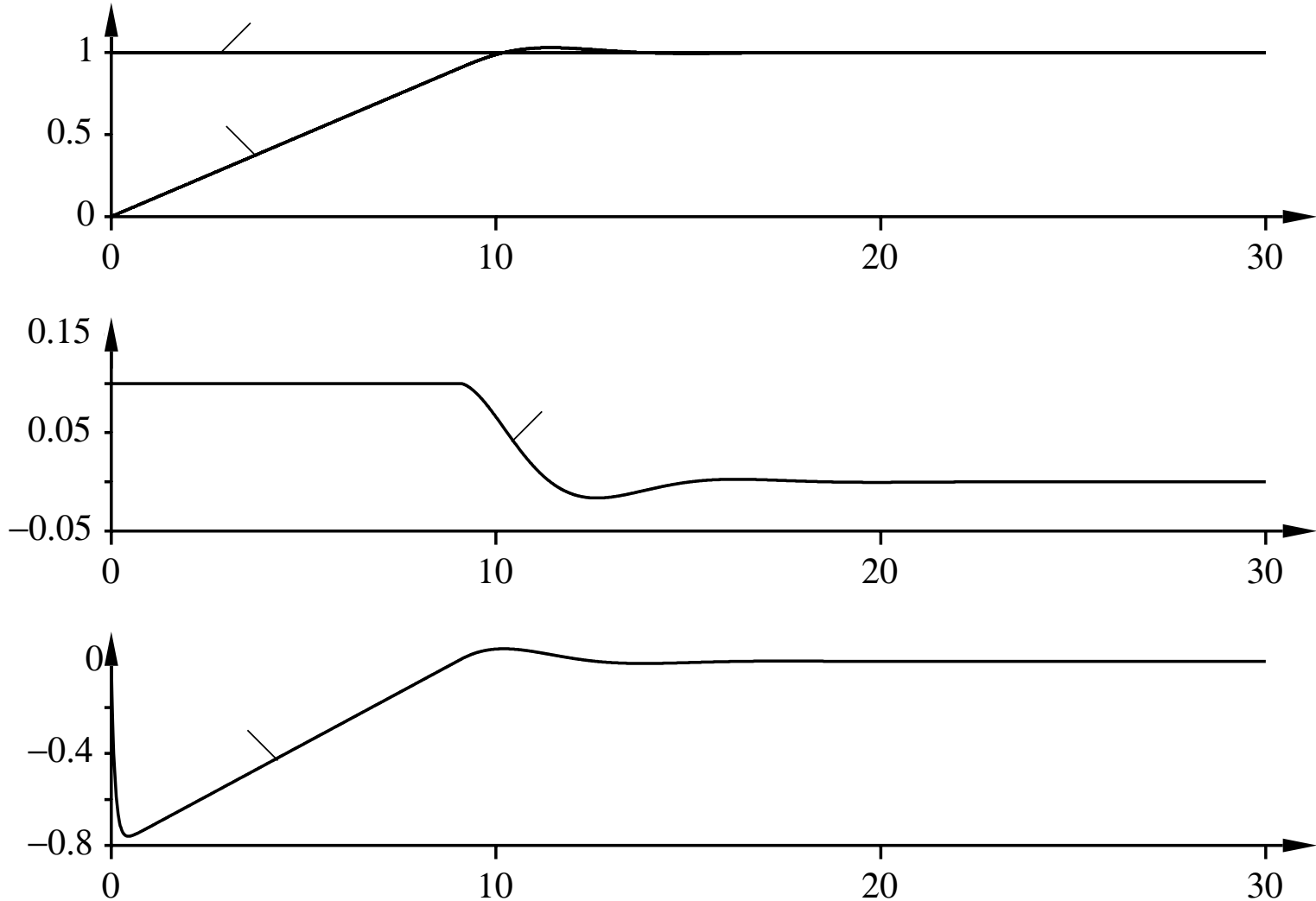
Tracking

- when the control signal saturates, the integral is recomputed so that its new value gives a control signal at the saturation limit
- to avoid resetting the integral due to, e.g., measurement noise, the recomputation is done dynamically, i.e., through a LP-filter with a time constant T_t .

Tracking



Tracking



Tuning

Parameters: $K, T_i, T_d, N, \gamma, T_t$

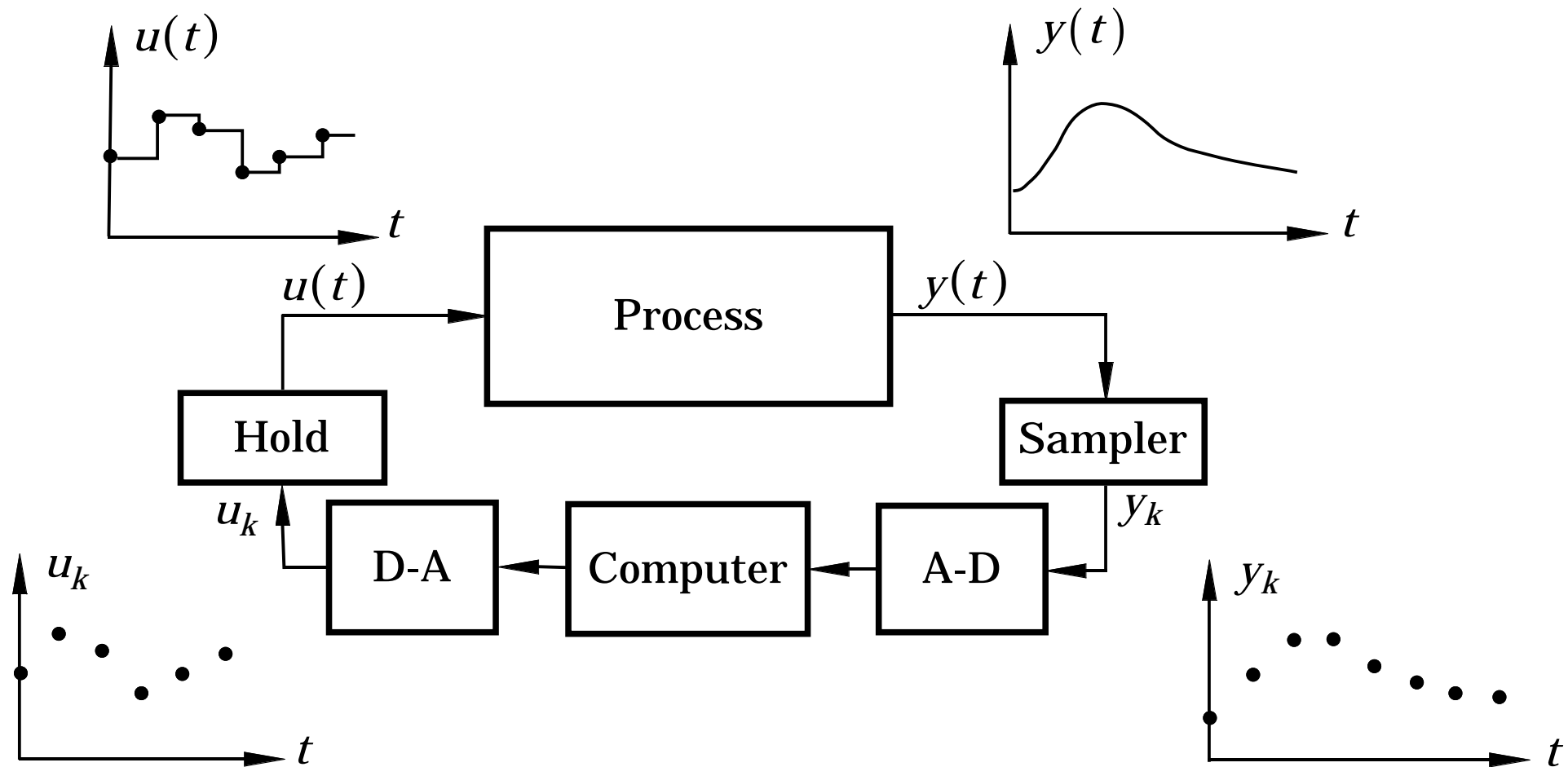
Methods:

- empirically, rules of thumb, tuning charts
- model-based tuning, e.g., pole-placement
- automatic tuning experiment
 - Ziegler-Nichols method
 - * step response method
 - * ultimate sensitivity method
 - relay method

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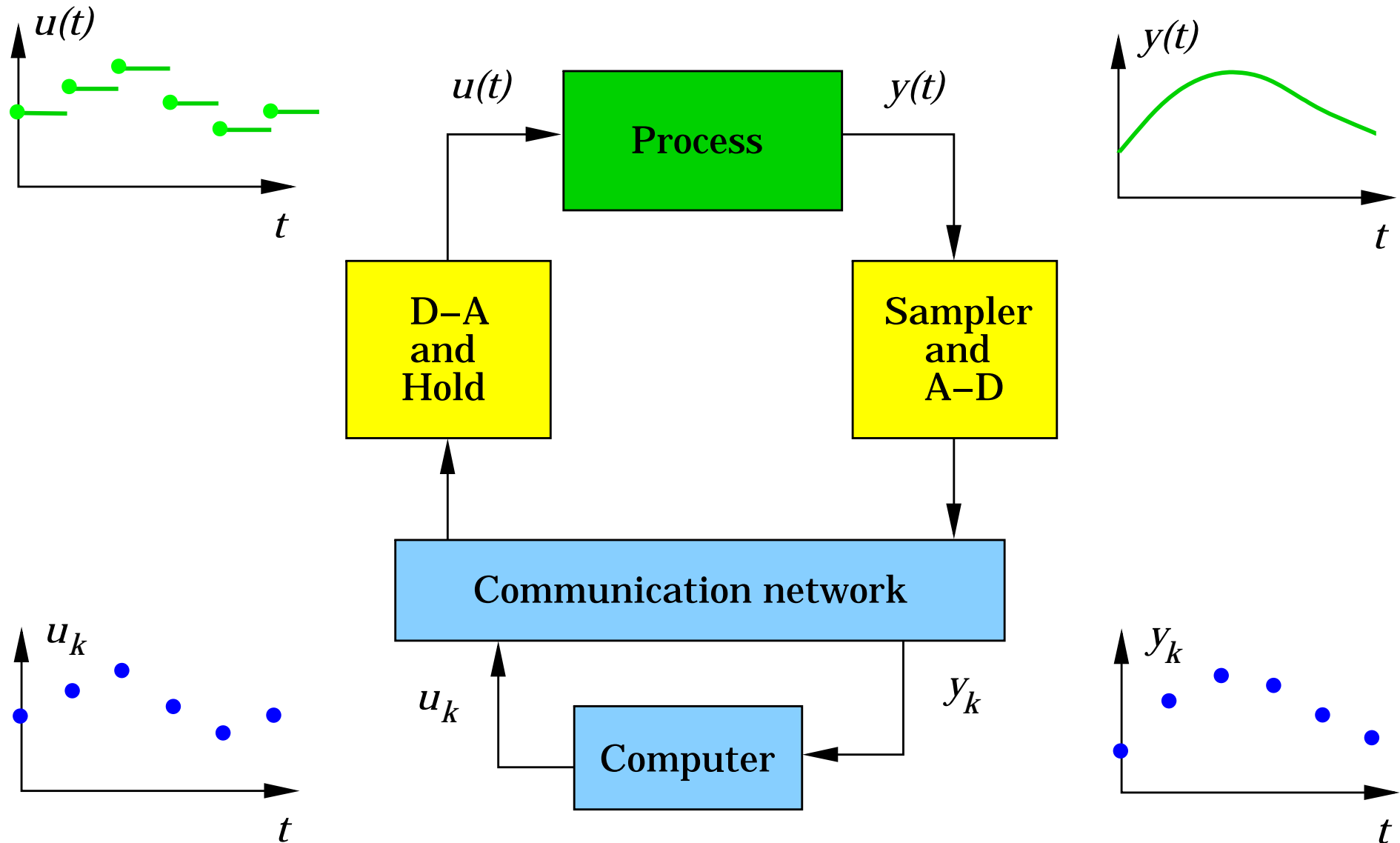
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Sampled-data control systems

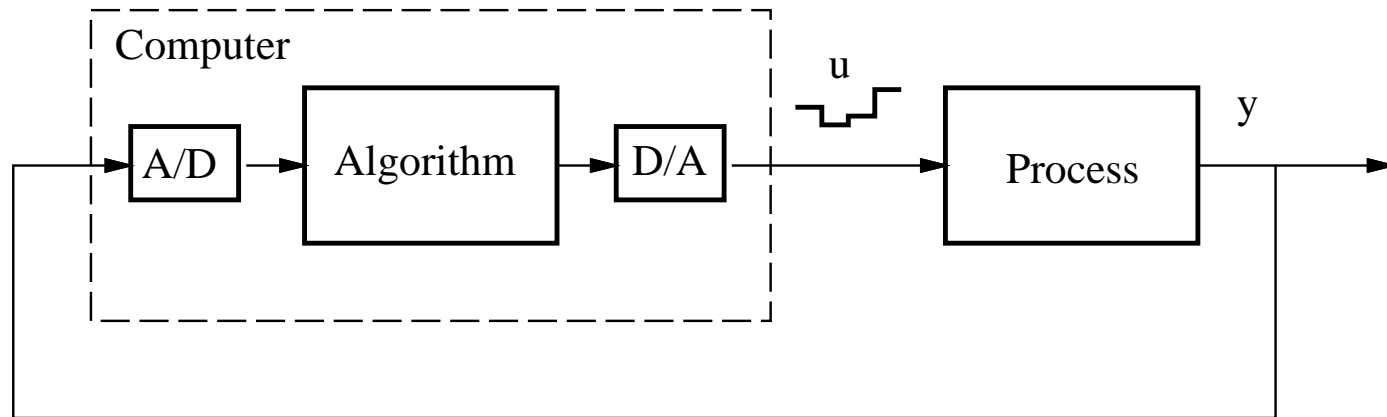


- Mix of continuous-time and discrete-time signals

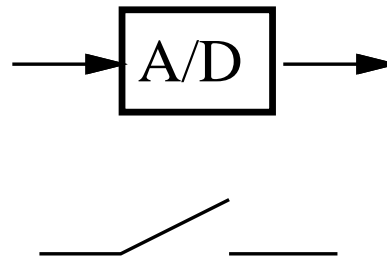
Networked control systems



Sampling



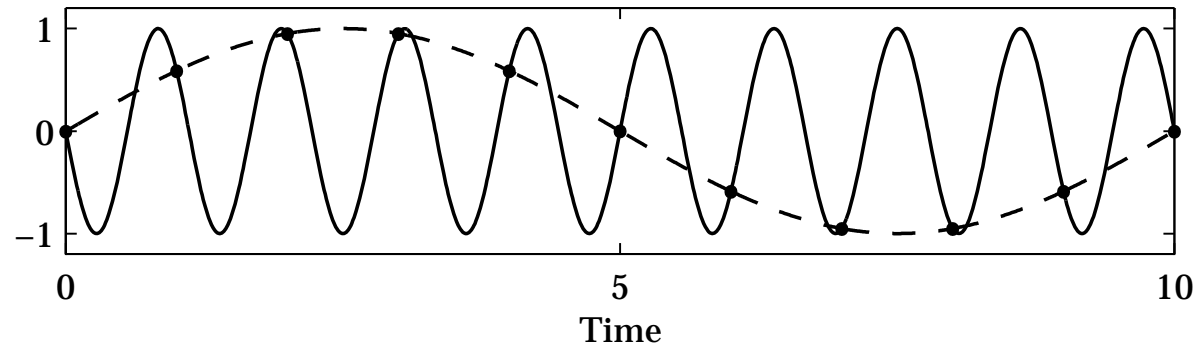
AD-converter acts as sampler



DA-converter acts as a hold device

Normally, zero-order-hold is used \Rightarrow piecewise constant control signals

Aliasing



$$\omega_s = \frac{2\pi}{h} = \text{sampling frequency}$$

$$\omega_N = \omega_s/2 = \text{Nyquist frequency}$$

Frequencies above the Nyquist frequency are folded and appear as low-frequency signals.

The fundamental alias frequency for a frequency f_1 is given by

$$f = |(f_1 + f_N) \bmod (f_s) - f_N|$$



$$f_1 = 0.9, f_s = 1, f_N = 0.5, f = 0.1$$

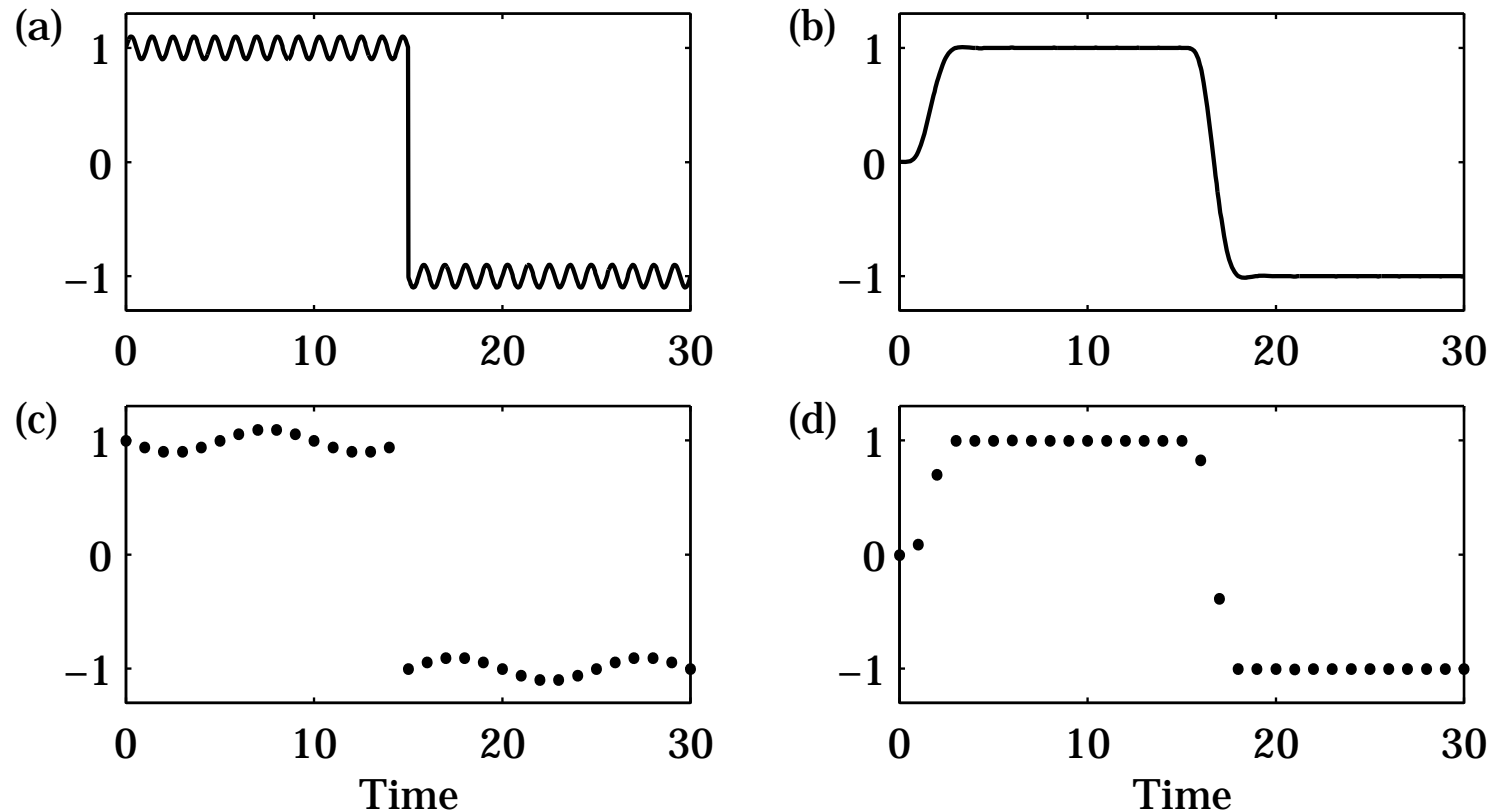
Anti-aliasing filter

Analog low-pass filter that eliminates all frequencies above the Nyquist frequency

- Analog filter
 - 2-6th order Bessel or Butterworth filter
 - Difficulties with changing h (sampling interval)
- Analog + digital filter
 - Fixed, fast sampling with fixed analog filter
 - Downsampling using digital LP-filter
 - Control algorithm at the lower rate
 - Easy to change sampling interval

The filter may have to be included in the control design

Example – Prefiltering



$$\omega_d = 0.9, \omega_N = 0.5, \omega_{alias} = 0.1$$

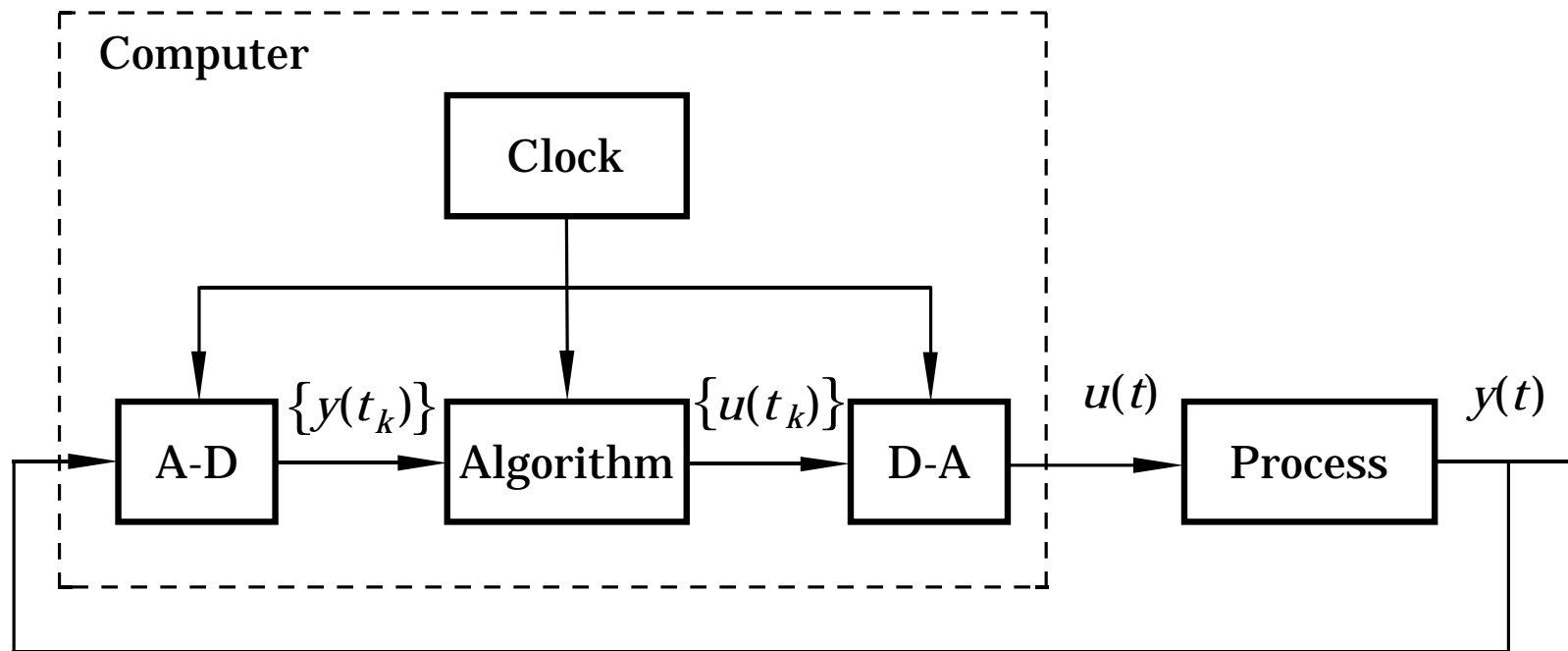
6th order Bessel with $\omega_B = 0.25$

Design approaches

Digital controllers can be designed in two different ways:

- Discrete-time design – sampled control theory
 - Sample the continuous system
 - Design a digital controller for the sampled system
 - * Z-transform domain
 - * state-space domain
- Continuous time design + discretization
 - Design a continuous controller for the continuous system
 - Approximate the continuous design
 - Use fast sampling

Sampled control theory



Basic idea: look at the sampling instances only

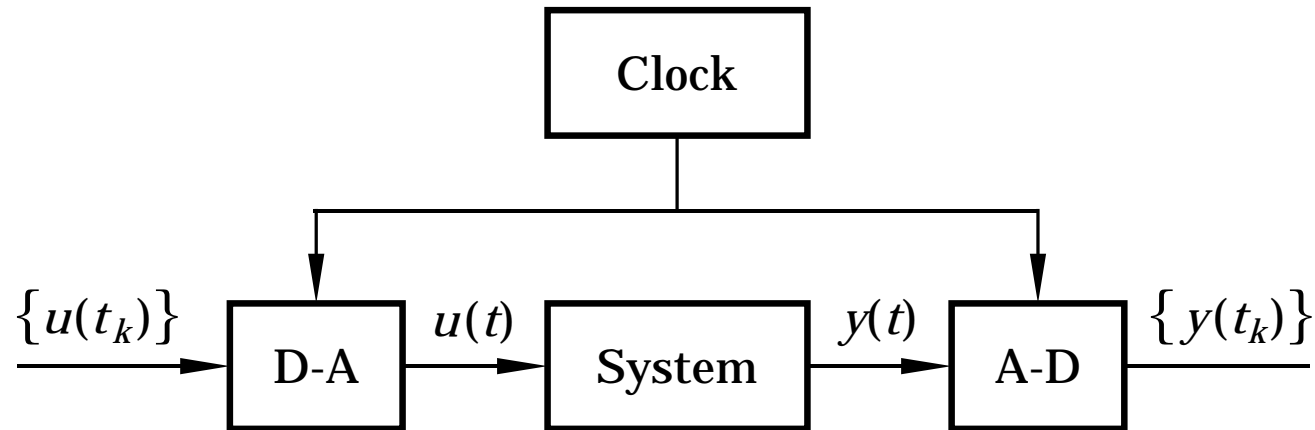
- System theory analogous to continuous-time systems
- Better performance can be achieved



Potential problem with intersample behaviour

Sampling of systems

Look at the system from the point of view of the computer



Zero-order-hold sampling of a system

- Let the inputs be piecewise constant
- Look at the sampling points only
- Solve the system equation

Sampling a continuous-time system

System description

$$\begin{aligned}\frac{dx}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Solve the system equation

$$\begin{aligned}x(t) &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^t e^{A(t-s')}Bu(s')ds' \\ &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^t e^{A(t-s')}ds' Bu(t_k) \quad (u \text{ const.}) \\ &= e^{A(t-t_k)}x(t_k) + \int_0^{t-t_k} e^{As}ds Bu(t_k) \quad (\text{variable change}) \\ &= \Phi(t, t_k)x(t_k) + \Gamma(t, t_k)u(t_k)\end{aligned}$$

Periodic sampling

Assume periodic sampling, i.e. $t_k = k \cdot h$, then

$$\begin{aligned}x(kh + h) &= \Phi x(kh) + \Gamma u(kh) \\y(kh) &= Cx(kh) + Du(kh)\end{aligned}$$

where

$$\begin{aligned}\Phi &= e^{Ah} \\ \Gamma &= \int_0^h e^{As} ds B\end{aligned}$$

Time-invariant linear system!

Example: Sampling of inverted pendulum

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

We get

$$\begin{aligned}\Phi &= e^{Ah} = \begin{pmatrix} \cosh h & \sinh h \\ \sinh h & \cosh h \end{pmatrix} \\ \Gamma &= \int_0^h \begin{pmatrix} \sinh s \\ \cosh s \end{pmatrix} ds = \begin{pmatrix} \cosh h - 1 \\ \sinh h \end{pmatrix}\end{aligned}$$

Several ways to calculate Φ and Γ . Matlab

Sampling a system with a time delay

Sampling the system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t - \tau), \quad \tau \leq h$$

we get the discrete-time system

$$x(kh + h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh - h)$$

where

$$\Phi = e^{Ah}$$

$$\Gamma_0 = \int_0^{h-\tau} e^{As} ds B$$

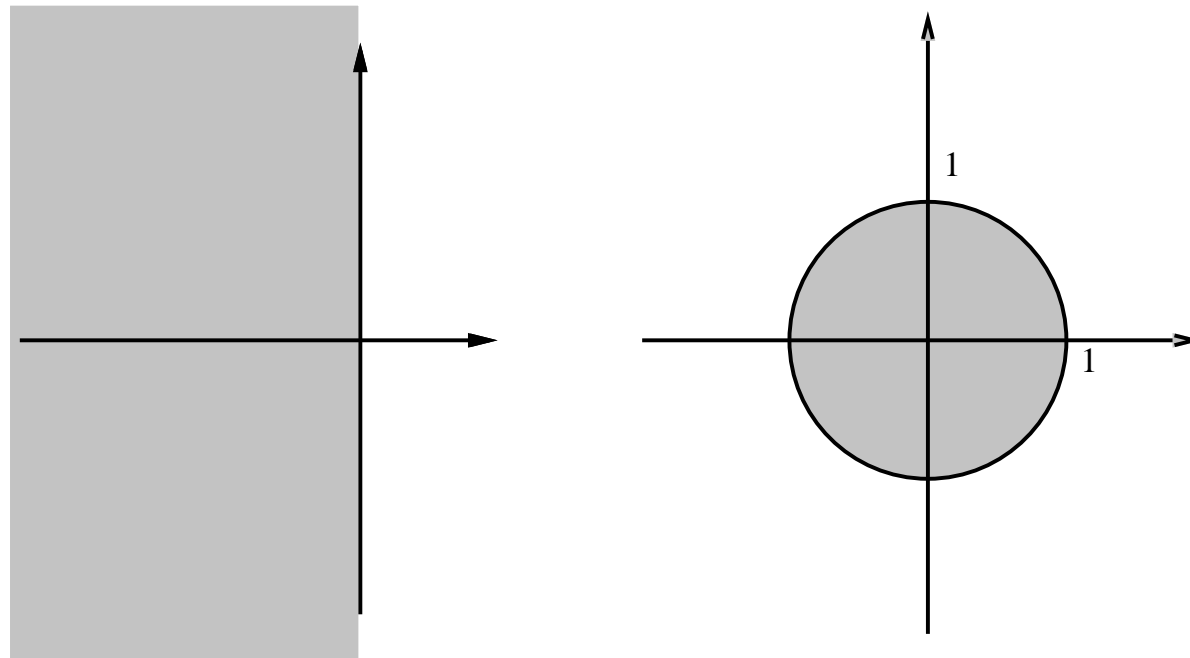
$$\Gamma_1 = e^{A(h-\tau)} \int_0^{\tau} e^{As} ds B$$

We get one extra state $(u(kh - h))$ in the sampled system



Stability region

- In continuous time the stability region is the complex left half plane, i.e., the system is stable if all the poles are in the left half plane.
- In discrete time the stability region is the unit circle.



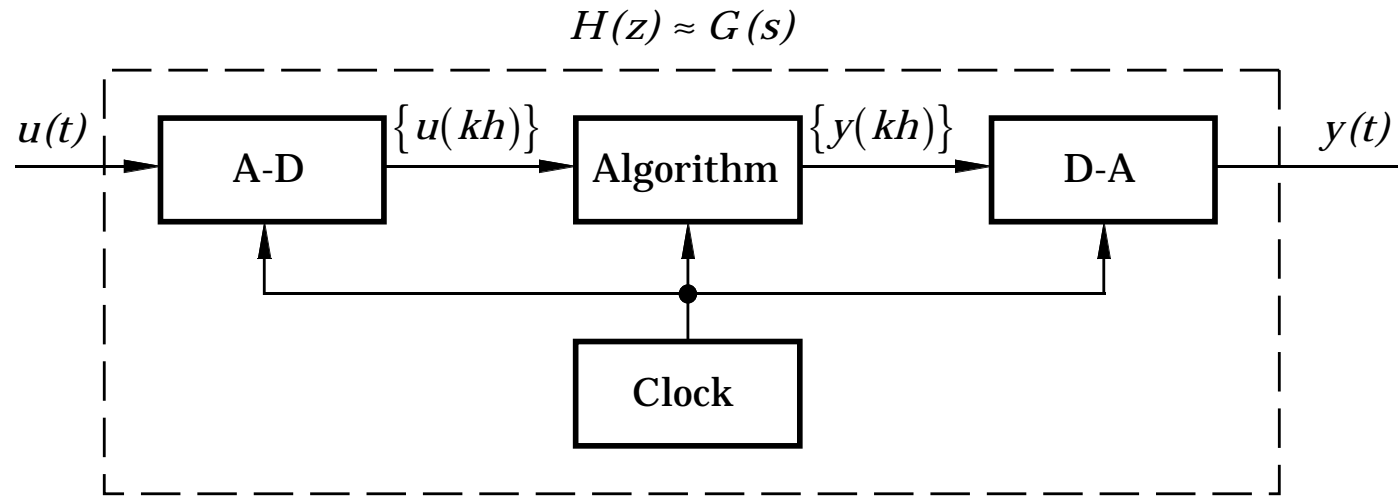
Digital control design

Similar to the continuous-time case, we can choose between

- frequency-domain design (loop shaping)
- pole-placement design
 - transfer function domain
 - state space domain
 - the poles are placed inside the unit circle
- optimal design methods (e.g. LQG)

Approximation of continuous-time design

Basic idea: Reuse the design



$G(s)$ is designed based on analog techniques

Want to get:

- $A/D + \text{Algorithm} + D/A \approx G(s)$

Methods:



approximate s , i.e., $H(z) = G(s')$

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- Other methods (Matlab)

Approximation methods

Forward Difference (Euler's method)

$$\frac{dx(t)}{dt} \approx \frac{x(t+h) - x(t)}{h}$$

$$s' = \frac{z-1}{h}$$

Backward Difference

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t-h)}{h}$$

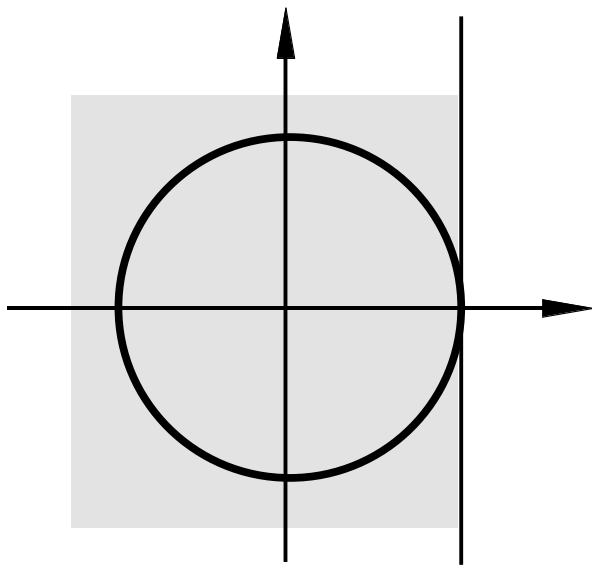
$$s' = \frac{z-1}{zh}$$

Tustin

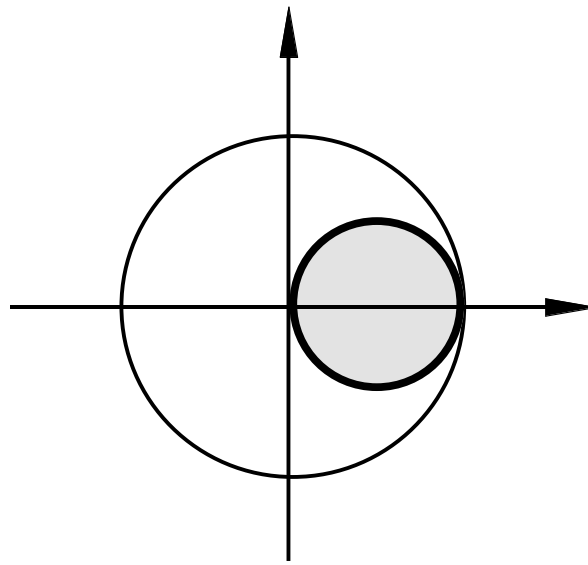
$$\frac{\frac{dx(t)}{dt} + \frac{dx(t+h)}{dt}}{2} \approx \frac{x(t+h) - x(t)}{h}$$
$$s' = \frac{2}{h} \frac{z-1}{z+1}$$

Stability of approximations

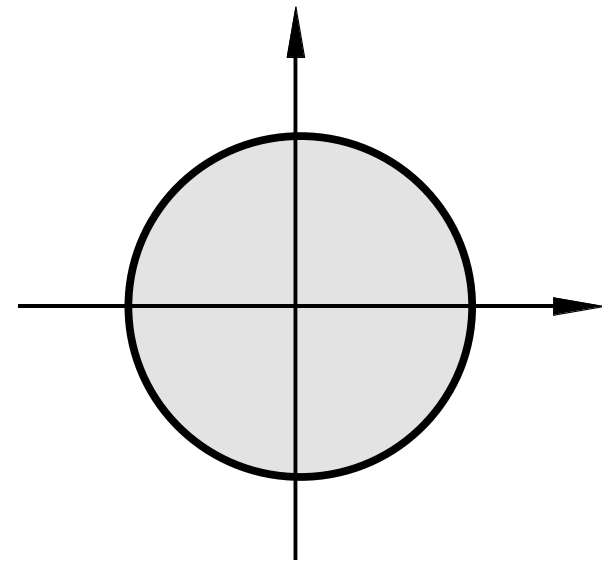
How is the continuous-time stability region (left half plane) mapped?



Forward differences



Backward differences



Tustin

Discretization of the PID controller

Continuous PID controller with $\gamma = 0$:

$$U(s) = K \left(R(s) - Y(s) \right) + \frac{1}{sT_i} \left(R(s) - Y(s) \right) - \frac{sT_d}{1 + sT_d/N} Y(s) \right)$$

Discretization

P-part:

$$P(k) = K(r(k) - y(k))$$

Discretization

I-part:

$$I(t) = \frac{K}{T_I} \int_0^t (r(\tau) - y(\tau)) d\tau$$
$$\frac{dI}{dt} = \frac{K}{T_I} (r(t) - y(t))$$

- Forward difference

$$\frac{I(k+1) - I(k)}{h} = \frac{K}{T_I} (r(k) - y(k))$$

$$I(k+1) := I(k) + (K \cdot h / T_I) \cdot (r(k) - y(k))$$

The I-part can be precalculated

- Backward difference



The I-part cannot be precalculated, $I(k) = f(r(k), y(k))$

Discretization

D-part (assume $\gamma = 0$):

$$D = K \frac{sT_D}{1 + sT_D/N} (-Y(s))$$

$$\frac{T_D}{N} \frac{dD}{dt} + D = -KT_D \frac{dy}{dt}$$

- Forward difference (unstable for small T_D)
- Backward difference

$$\frac{T_D}{N} \frac{D(k) - D(k-1)}{h} + D(k) = -KT_D \frac{y(k) - y(k-1)}{h}$$

$$D(k) = \frac{T_D}{T_D + Nh} D(k-1) - \frac{KT_D N}{T_D + Nh} (y(k) - y(k-1))$$

Discretization

Tracking:

$v := P + I + D;$

$u := \text{sat}(v, u_{\max}, u_{\min});$

$I := I + (K \cdot h / T_i) \cdot (r - y) + (h / T_t) \cdot (u - v);$

PID code

PID-controller with anti-windup ($\gamma = 0$).

```
r = ref.get();  
y = yIn.get();  
D = ad * D - bd * (y - yold);  
v = K*(r - y) + I + D;  
u = sat(v,umax,umin);  
uOut.put(u);  
I = I + (K*h/Ti)*(r - y) + (h/Tt)*(u - v);  
yold = y;
```

ad and bd are precalculated parameters given by the backward difference approximation of the D-term.

Further Reading

- B. Wittenmark, K. J. Åström, K.-E. Årzén: “Computer Control: An Overview.” IFAC Professional Brief, 2002. (93 pages, available at <http://www.control.lth.se>)
- K. J. Åström, Tore Hägglund: “Advanced PID Control.” The Instrumentation, Systems, and Automation Society, 2005.
- A. Cervin: “Integrated Control and Real-Time Scheduling.” PhD Thesis, Lund University, 2003. (Available at <http://www.control.lth.se>)