Finite-State Systems

Gérard Berry

http://www-sop.inria.fr/members/Gerard.Berry/

Collège de France
Informatics and Digital Sciences Chair

INRIA Sophia-Antipolis

Edinburgh Informatics Forum, September 30th, 2010

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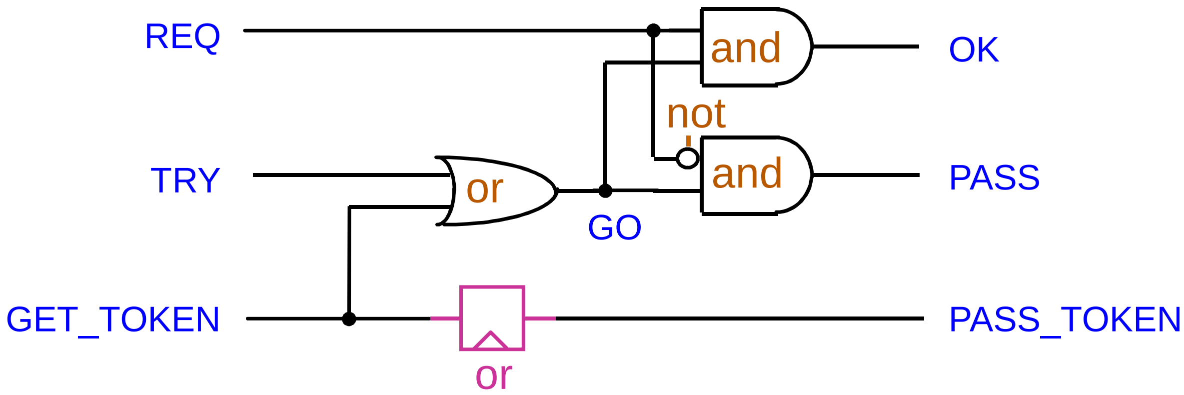
Finite-State Systems

- General principles :
  - the system only uses finite resources
  - it proceeds by a sequence of elementary transitions
- Multiple formalisms
  - machines: finite automata, Boolean circuits
  - regular expressions: shell, lex, perl, etc.
  - hierarchical state machines: Statecharts , SyncCharts
  - synchronous languages : Esterel, Lustre, Signal
  - semantics : regular languages

Application Fields

- Language analysis
  - natural languages
  - programming languages
- Electronic circuits
  - data path / control path
  - memory / cache handling
  - NoCs (Networks on Chips), USB, SATA, etc.
- Communication protocols
  - inititiation and maintenance of communication links
  - error detection and handling, packet retransmission
- Industry
  - real-time control, programmable logic controllers (PLCs)

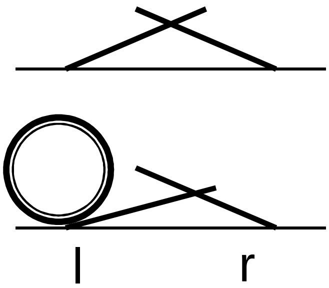
Boolean Circuits



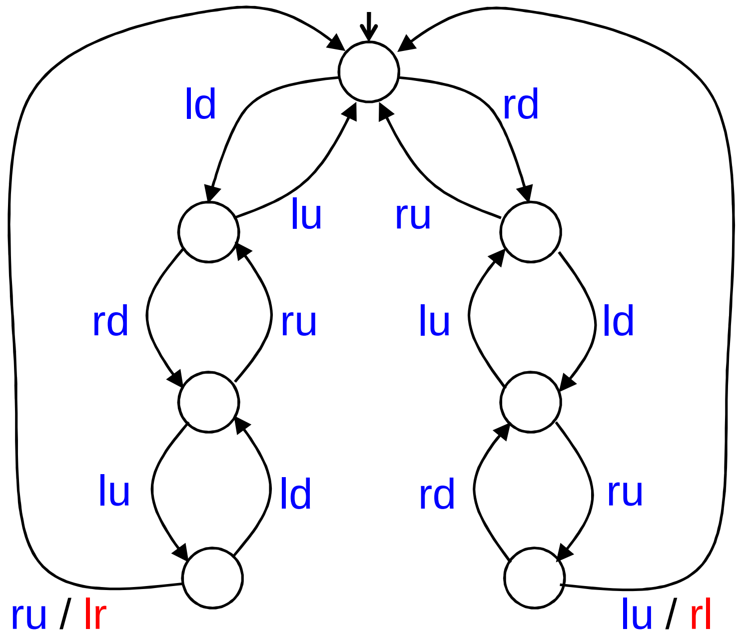
OK = REQ and GO
PASS = not REQ and GO
GO = TRY or GET_TOKEN
PASS_TOKEN = reg(GET_TOKEN)

Automata : Explicit State Machines

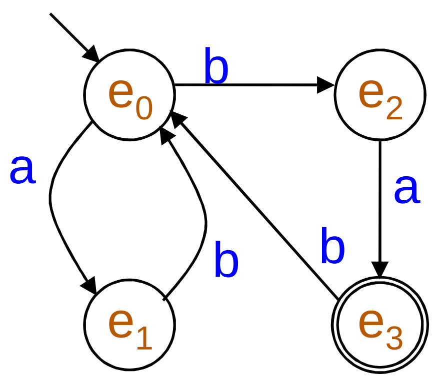
axle counter



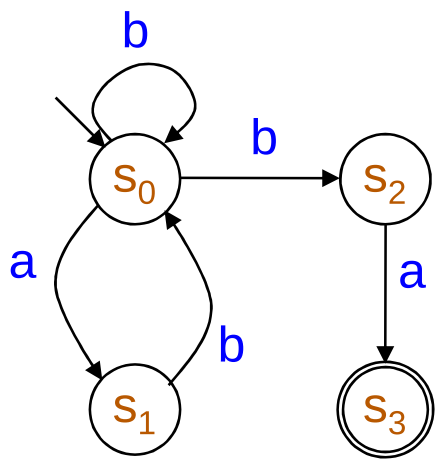
- inputs:
  ld, lu, rd, ru
- outputs:
  → lr, ← rl



Recognizer Automata



Deterministic



Non-deterministic

(w.r.t. s_0 and b)

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Agenda

1. Regular Expressions
2. Derivatives and Translation to Automata
3. The Berry-Sethi Algorithm
4. Determinization and minimization
5. From Automata to Boolean Circuits
6. Automatic Sequences and Transcendental Numbers

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Regular Languages

- Alphabet : finite set of letters a, b, c, x, y...
- Word: finite sequence u, v, w of letters

  ε = empty word
- Concatenation of words:

  $ab \cdot cd = abcd$
- Language L: set of words
- Regular languages: the smallest family of languages containing the finite languages and closed by concatenation and union

Regular Expressions

- 0 — $L(0) = \emptyset$
- 1 — $L(1) = \{\varepsilon\}$
- a — $L(a) = \{a\}$
- $e+e'$ — $L(e+e') = L(e) \cup L(e')$
- $e \cdot e'$ or $e\,e'$ — $L(e \cdot e') = \{uu' \mid u \in L(e),\ u' \in L(e')\}$
- e^* — $L(e^*) = \{u_0 u_1 \ldots u_n \mid n \geq 0,\ u_i \in L(e)\}$

$$e+e' = e'+e$$
$$e+(e'+e'') = (e+e')+e''$$
$$e \cdot (e'+e'') = e \cdot e' + e \cdot e''$$
$$(e+e') \cdot e'' = e \cdot e'' + e' \cdot e''$$

$$0+e = e+0 = e$$
$$0 \cdot e = e \cdot 0 = 0$$
$$1 \cdot e = e \cdot 1 = 1$$

Test Expression: (ab+b)* ba

ba

abba

bba

abbabba

bbbbababbbbba

...

Test Expression: (ab+b)* ba

ba

abba

bba

abbabba

bbbbababbbbba

...

Test for Empty Word Generation

- $\varepsilon(L) = 1$ if $\varepsilon \in L$
  $= 0$ otherwise

- $\varepsilon(0) = 0$
- $\varepsilon(1) = 1$
- $\varepsilon(a) = 0$
- $\varepsilon(e+e') = \varepsilon(e) + \varepsilon(e')$
- $\varepsilon(e \cdot e') = \varepsilon(e) \cdot \varepsilon(e')$
- $\varepsilon(e^*) = 1$

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Derivatives of Regular Languages

$L \rightarrow$ ba, abba, bba, abbabba, bbbbababbbbba,...

$a^{-1}(L) \rightarrow$ ba, abba, bba, abbabba, bbbbababbbbba,...

$b^{-1}(L) \rightarrow$ ba, abba, bba, abbabba, bbbbababbbbba,...

$u^{-1}(L) = \{ v \mid uv \in L \}$ $ua^{-1}(L) = a^{-1}(u^{-1}(L))$

what remains to be written after writing u

Derivatives of Regular Expressions

$a^{-1}(e)$ = regular expression generating $a^{-1}(L(e))$

- $a^{-1}(0) = 0$
- $a^{-1}(1) = 0$
- $a^{-1}(a) = 1$
- $a^{-1}(b) = 0$ si $b \neq a$
- $a^{-1}(e + e') = a^{-1}(e) + a^{-1}(e')$
- $a^{-1}(e \cdot e') = a^{-1}(e) \cdot e' + \varepsilon(e) \cdot a^{-1}(e')$
- $a^{-1}(e^*) = a^{-1}(e) \cdot e^*$

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Test Expression: $(ab+b)^*\ ba$

- $a^{-1} \rightarrow$ ba, abba, bba, abbabba, bbbbababbbbba,...
- $a^{-1}((ab+b)^* ba) = a^{-1}((ab+b)^*) \cdot ba$
  $+ \varepsilon((ab+b)^*) \cdot a^{-1}(ba)$
  $= a^{-1}(ab+b) \cdot (ab+b)^* \cdot ba + 0$
  $= b \cdot (ab+b)^* \cdot ba$ OK

- $b^{-1} \rightarrow$ ba, abba, bba, abbabba, bbbbababbbbba,...
- $b^{-1}((ab+b)^* ba) = b^{-1}((ab+b)^*) \cdot ba$
  $+ \varepsilon((ab+b)^*) \cdot b^{-1}(ba)$
  $= b^{-1}(ab+b) \cdot (ab+b)^* \cdot ba$
  $+ a$
  $= (ab+b)^* ba + a$ OK

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From Derivatives to Automata

Theorem (Brzozowski) : a regular expression e has at most a finite number of distinct (simplified) derivatives $u^{-1}(e)$

Corollary : one can construct a deterministic finite automaton recognizing L(e) as follows:

- the states are the distinct derivatives of e
- the initial state is e
- there is a-labeled transition from $u^{-1}(e)$ to $ua^{-1}(e)$ if both derivatives are nonempty
- a state e' is final if ε(e') = 1

Convergent Iterative Process

- $e = e_0 = (ab+b)^*ba$
- $a^{-1}(e_0) = b(ab+b)^*ba = e_1$
- $b^{-1}(e_0) = (ab+b)^*ba + a = e_2$
- $a^{-1}(e_1) = 0$
- $b^{-1}(e_1) = (ab+b)^*ba = e_0$
- $a^{-1}(e_2) = b(ab+b)^*ba + 1 = e_3$
- $b^{-1}(e_2) = (ab+b)^*ba + a = e_2$
- $a^{-1}(e_3) = 0$
- $b^{-1}(e_3) = (ab+b)^*ba = e_0$

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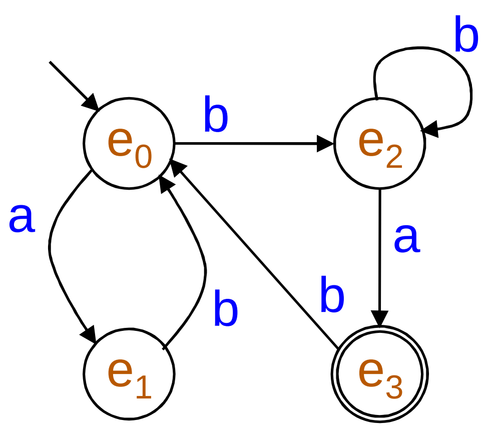
Constructing the Deterministic Automaton

- $a^{-1}(e_0) = e_1$
- $b^{-1}(e_0) = e_2$
- $a^{-1}(e_1) = 0$
- $b^{-1}(e_1) = e_0$
- $a^{-1}(e_2) = e_3$
- $b^{-1}(e_2) = e_2$
- $a^{-1}(e_3) = 0$
- $b^{-1}(e_3) = e_0$
- $\varepsilon(e_3) = 1$



deterministic automaton
(Brzozoswki)

Other Results

Corollary : the class of regular languages is closed by complementation and intersection

proof: for negation, revert terminal and non-terminal states
intersection implied by union and negation

Remark: derivation works with negation and intersection operators in expressions (but explosive):

$a^{-1}(\neg e) = \neg a^{-1}(e) \qquad a^{-1}(e \cap e') = a^{-1}(e) \cap a^{-1}(e')$

Reciprocal theorem: every language recognized by an automaton is regular

proof : iteratively construct a regular expression by transitive closure of a state-indexed square matrix of expressions, starting from automata transitions

Exponential Explosion Danger Both Ways !

- The Brzozowski finite automation may be exponentially bigger than e
- For some expressions, this is true for all equivalent deterministic automata
- Conversely, for some automata, any equivalent regular expression may be exponentially bigger

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Linear Expression

- an expression is linear if it contains each letter at most once
- linearize expressions by uniquely indexing letters

$s_0 = (a_0 b_1 + b_2)^* b_3 a_4$

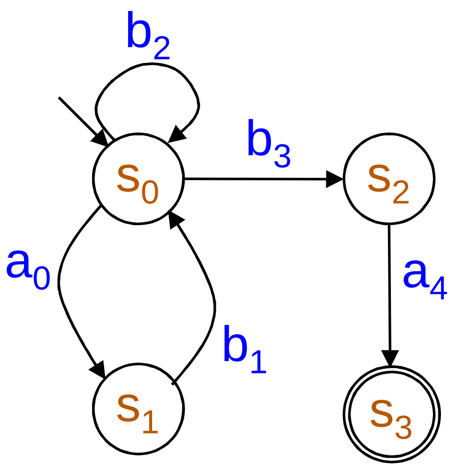
$a_0^{-1}(s_0) = b_1(a_0 b_1 + b_2)^* b_3 a_4 = s_1$

$b_2^{-1}(s_0) = (a_0 b_1 + b_2)^* b_3 a_4 = s_0$

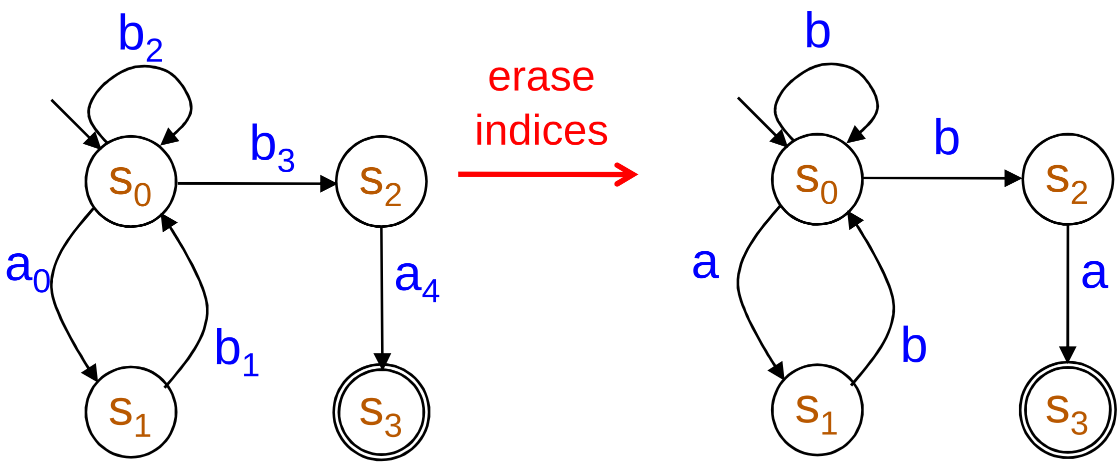
$b_3^{-1}(s_0) = a_4 = s_2$

$b_1^{-1}(s_1) = s_0$

$a_4^{-1}(s_0) = \varepsilon = s_3$



Non-Deterministic Automaton



Non-deterministic automaton recognizing $L(e_0)$

Linear Derivatives Are Simple

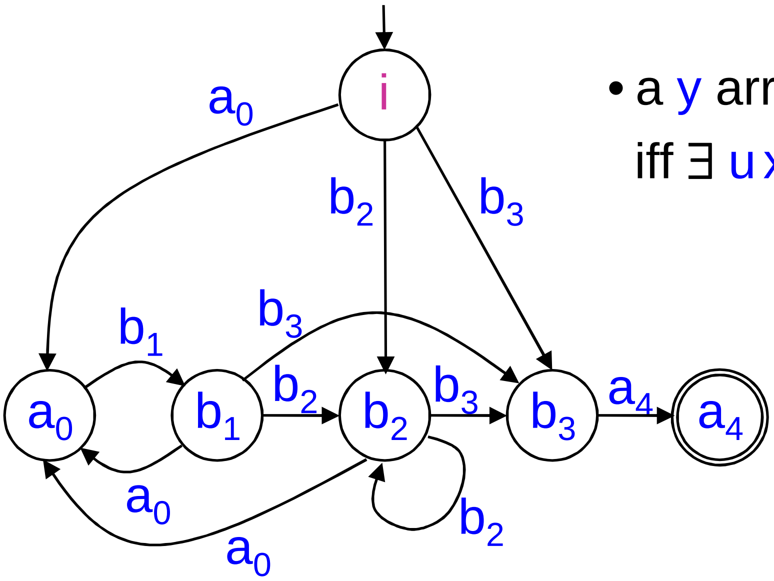
Theorem : if e is linear, then a non-null derivative is fully characterized by its last letter : all derivatives of the form $ux^{-1}(e)$ are null or equal

Proof : by induction on $|e|$ (size of e). Assume $ux^{-1}(e)$ non-null.

1. obvious if $e = x$ (then $|u| = 0$)
2. if $e = e_1 + e_2$, then x appears either in e_1 or in e_2 but not in both
   - if x appears in e_1, then $ux^{-1}(e) = ux^{-1}(e_1)$ with $|e_1| < |e|$
   - if x appears in e_2, then $ux^{-1}(e) = ux^{-1}(e_2)$ with $|e_2| < |e|$
3. if $e = e_1 \cdot e_2$, similar proof by computing $ux^{-1}(e_1 \cdot e_2)$ as a sum of two terms such that only one can contain x
4. if $e = e_1^*$, similar proof to case 3

The Automaton of a Linear Expression

$s_0 = (a_0 b_1 + b_2)^* b_3 a_4$



- an arrow from i to x if $\exists\, u = xu \in L(s_0)$
- a y arrow from x to y iff $\exists\, u\,xy\,v \in L(s_0)$

The Berry-Sethi Algorithm

Building a non-deterministic automaton for e linear :

1. build an initial state plus a state per letter x, representing $ux^{-1}(e)$ for all u
2. add an arrow labeled x from the initial state to any state x such that x can appear as first letter, i.e., $x^{-1}(e) \neq 0$
3. build an arrow labeled y from state x to state y iff $ux^{-1}(e) \neq 0$ and $uxy^{-1}(e) \neq 0$, i.e. there exists $uxyv \in L(e)$, i.e. iff y may follow x in $L(e)$
4. set state x terminal if there exists a word $ux \in L(e)$

Computing first et last

- first (0) = first(1) = ∅
- first (a) = {a}
- first(e+e') = first(e) ∪ first(e')
- first(e·e') = first(e) if ε(e) = 0
  = first(e) ∪ first(e') otherwise
- first(e*) = first(e)

first letters that e can write

- last (0) = last(1) = ∅
- last (a) = {a}
- last(e+e') = last(e) ∪ last(e')
- last(e·e') = last(e') if ε(e')=0
  = last(e) ∪ last(e') otherwise
- last(e*) = last(e)

last letters that e can write

Computing whether y can follow x

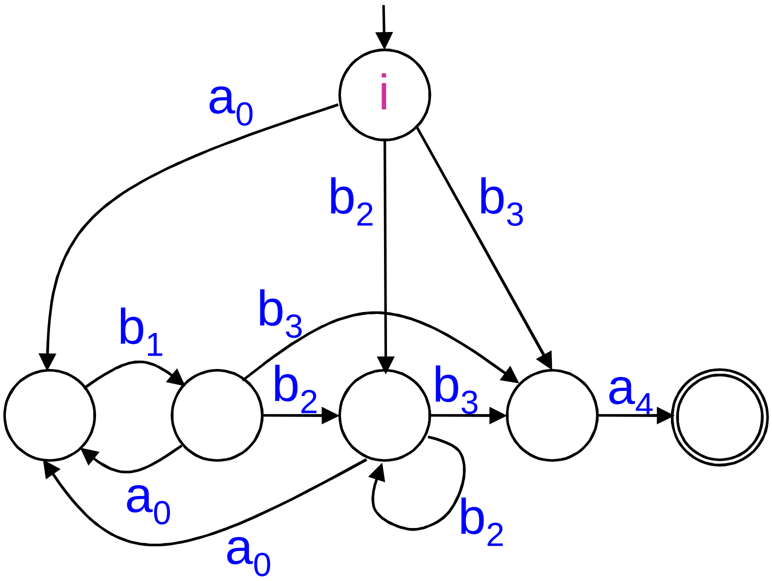
follow(e, x, y) = 1 iff $\exists$ w x y w' $\in$ L(e)

- follow(0, x, y) = follow(1, x, y) = follow(a, x, y) = 0
- follow(e+e', x, y) = follow(e, x, y) $\vee$ follow(e', x, y)
- follow(e · e', x, y) = follow(e, x, y)
  + follow(e', x, y)
  + (x $\in$ last(e) · y $\in$ first(e'))
- follow(e*, x, y) = follow(e, x, y)
  + (x $\in$ last(e) · y $\in$ first(e))

Exercice : find more efficient formulae

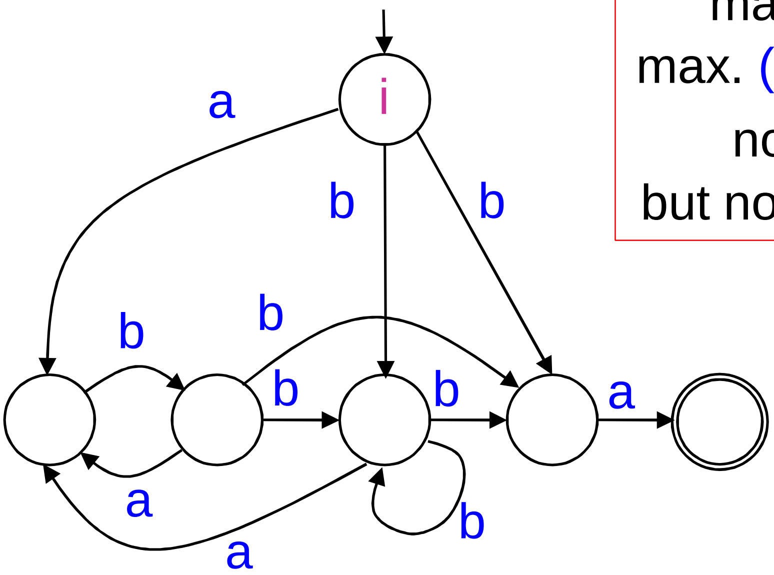
The Berry-Sethi Algorithm

$s_0 = (a_0 b_1 + b_2)^* b_3 a_4$



Erasing Indices

e = (ab+b)*ba



for n letter occurrences
max. n+1 states
max. $(n+1)^2$ transitions

no explosion !
but non-deterministic..

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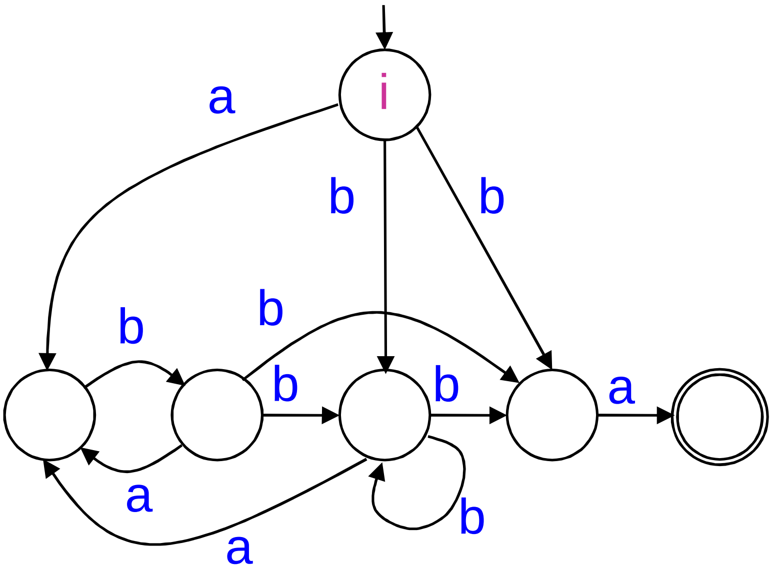
Optimizing Away the Initial State

e = (ab+b)*ba



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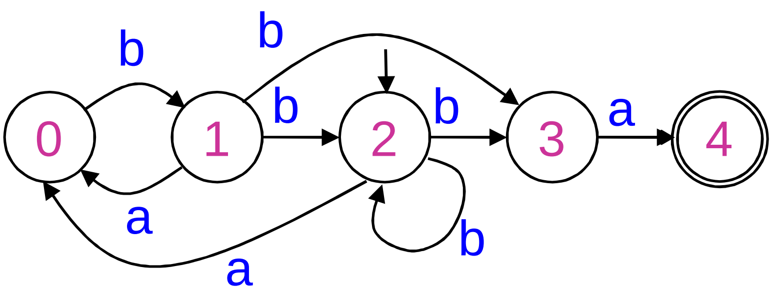
Determinization

e = (ab+b)*ba

00100 : initial

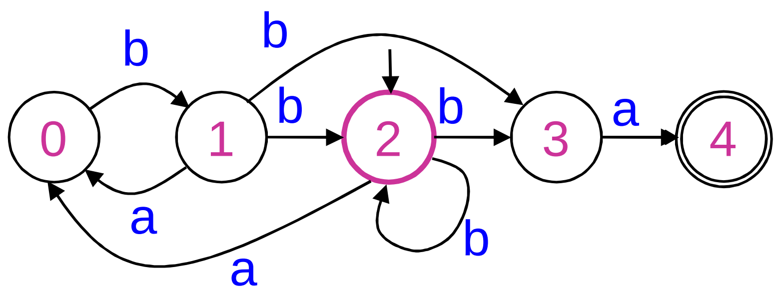


Determinization

$e = (ab+b)^*ba$

00100 : initial

00100 : a →

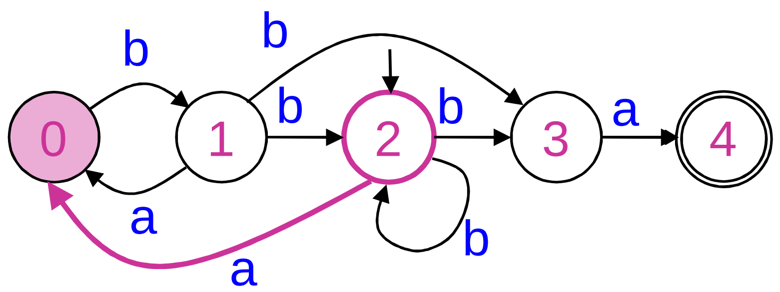


Determinization

00100 : a → 10000

e = (ab+b)*ba

00100 : initial

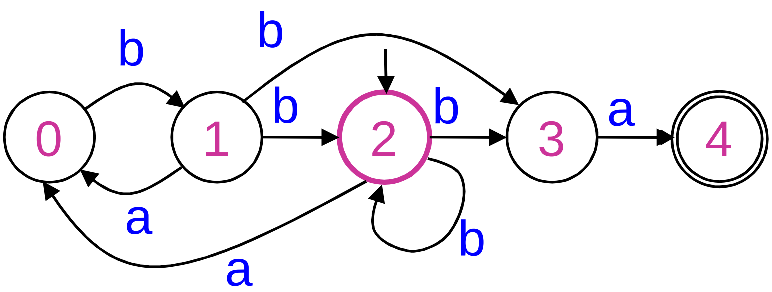


Determinization

$e = (ab+b)^{*}ba$

00100 : initial

00100 : a → 10000
b →

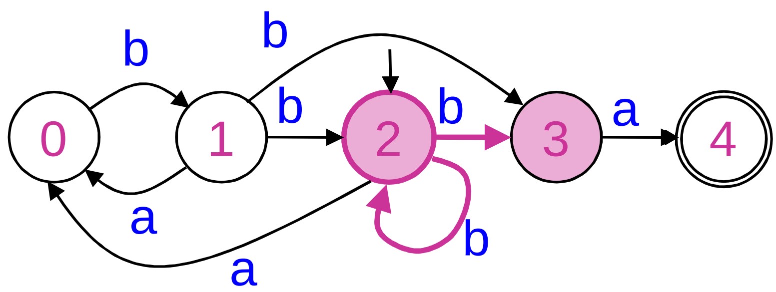


Determinization

$e = (ab+b)^*ba$

00100 : initial

00100 : a → 10000
b → 00110



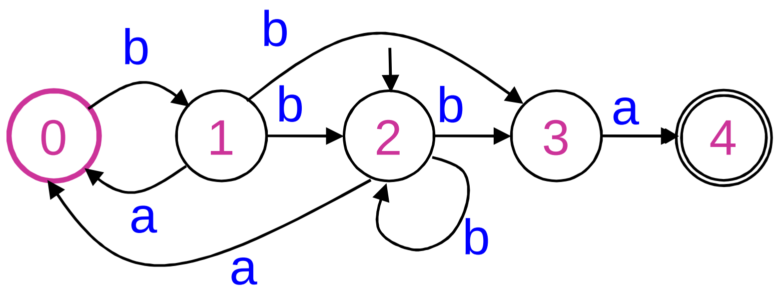
Determinization

e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b →



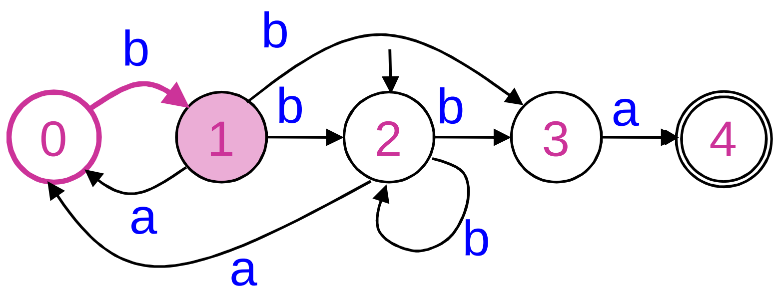
Determinization

e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000



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Determinization

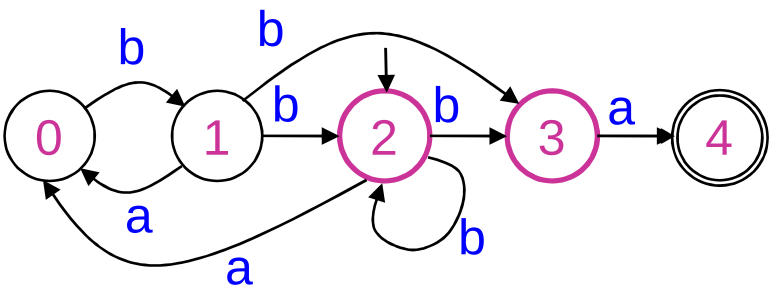
e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a →



Determinization

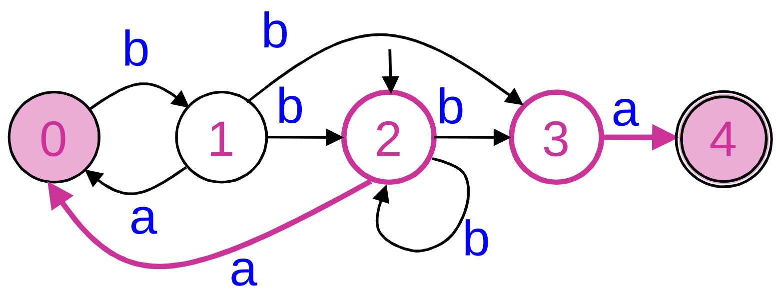
e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → 10001



Determinization

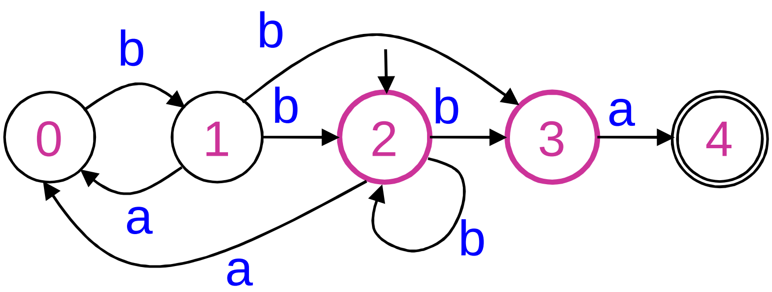
e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → <u>10001</u>
b →



Determinization

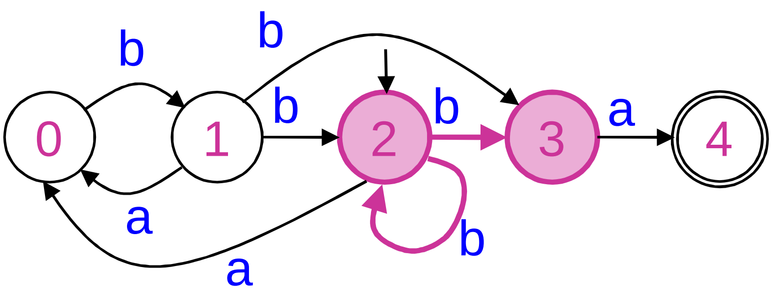
e = (ab+b)*ba

00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → 10001
b → 00110



Determinization

$e = (ab+b)^{*}ba$

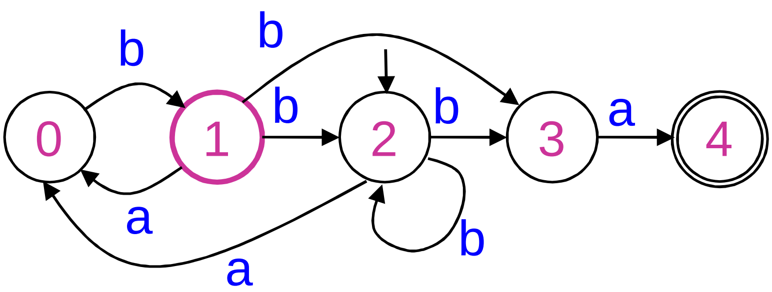
00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → <u>10001</u>
b → 00110

01000 : a



Determinization

e = (ab+b)*ba

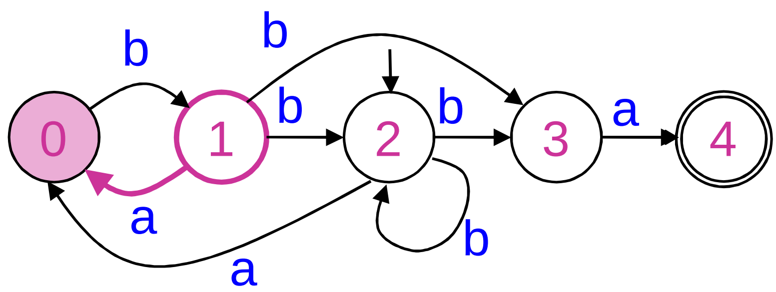
00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → 10001
b → 00110

01000 : a → 10000



Determinization

e = (ab+b)*ba

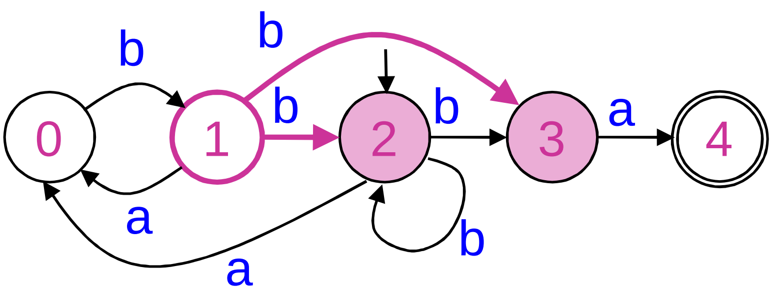
00100 : initial

00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → 10001
b → 00110

01000 : a → 10000
b → 00110



Determinization

$e = (ab+b)^{*}ba$

00100 : initial

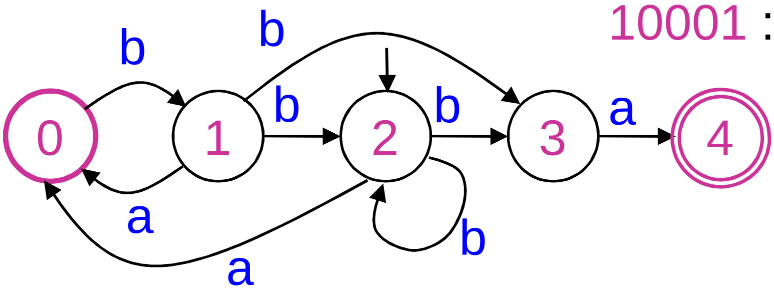
00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → <u>10001</u>
b → 00110

01000 : a → 10000
b → 00110

10001 : b



Determinization

e = (ab+b)*ba

00100 : initial

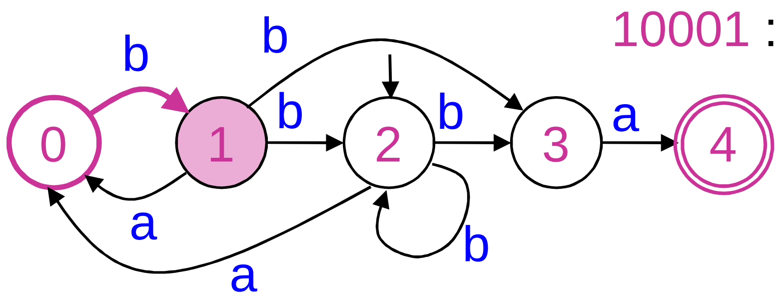
00100 : a → 10000
b → 00110

10000 : b → 01000

00110 : a → <u>10001</u>
b → 00110

01000 : a → 10000
b → 00110

10001 : b → 01000



Determinization

$e = (ab+b)^*ba$

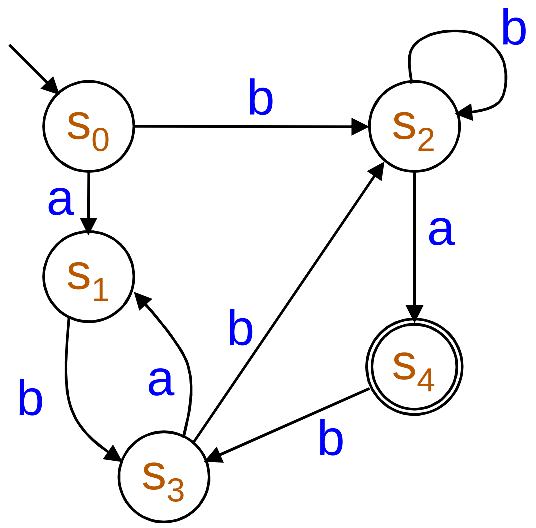
$s_0 = 00100$

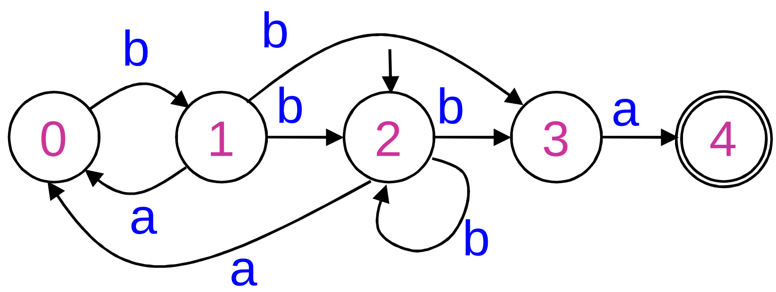
$s_1 = 10000$

$s_2 = 00110$

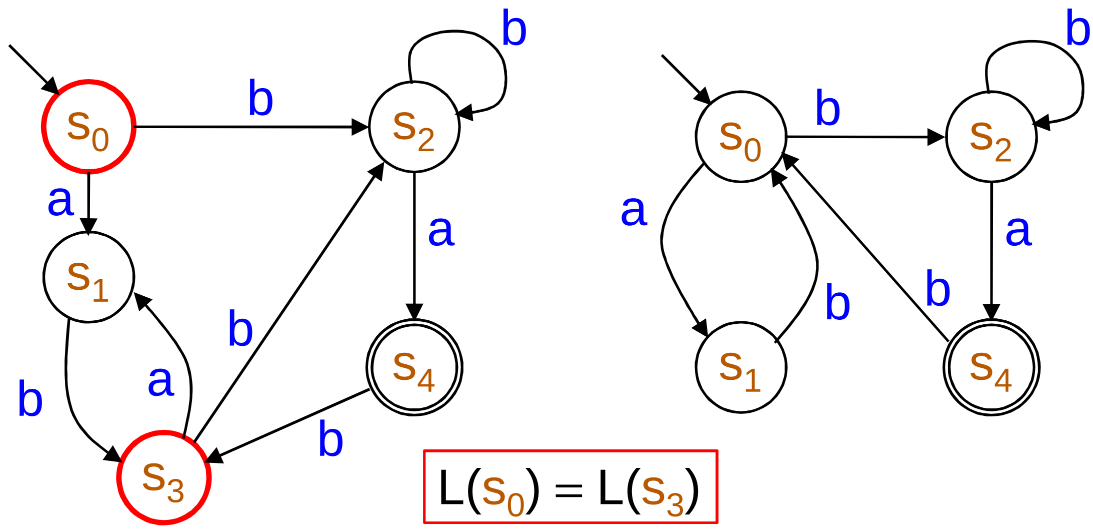
$s_3 = 01000$

$s_4 = 10001$





Minimizing Deterministic Automata



- Theorem : for any regular language L, there exists a minimal deterministic automaton recognizing L.
  Idea : states= nonempty $u^{-1}(L)$
- Bonus : there exists a fast minimization algorithm

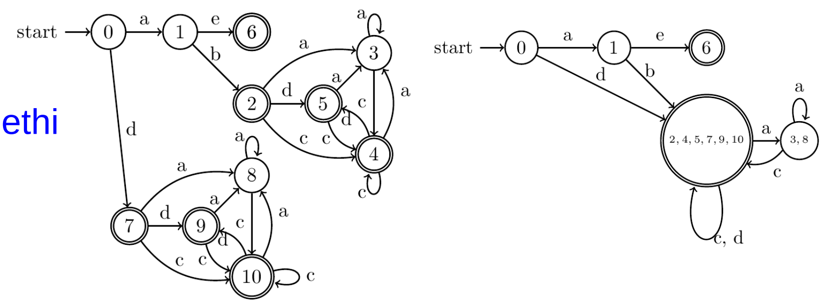
Summary

- A regular language is generated (or recognized) by a regular expression or by a finite automaton
- One can construct a deterministic automaton from a regular explosion, but risking an exponential explosion
- One can efficiently construct a non-deterministic aotomaton of maximal size n^2 for an expression of size n
- On can determinize this automaton, but with potential exponential explosion
- One can efficiently minimize a deterministic automaton (unless one has exploded beforehand)

Other Results

- In general, there is no minimal non-deterministic automaton for a given language
- One can improve upon the Berry-Sethi construction (cf. Antimirov, Champarnaud, Razet)

Berry-Sethi



Antimirov Champarnaud-Ziadi

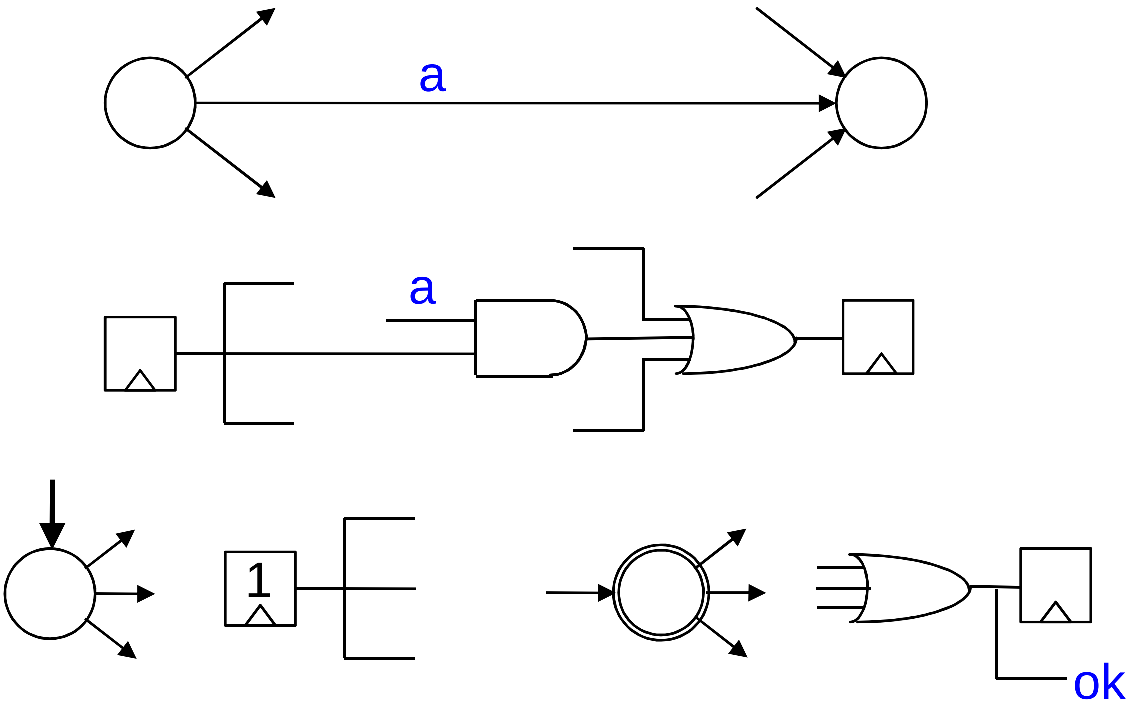
$$E = a(b(a^*c + d)^* + e) + d(a^*c + d)^*$$

G. Berry, L3, Edinburgh, 30/09/2010 55

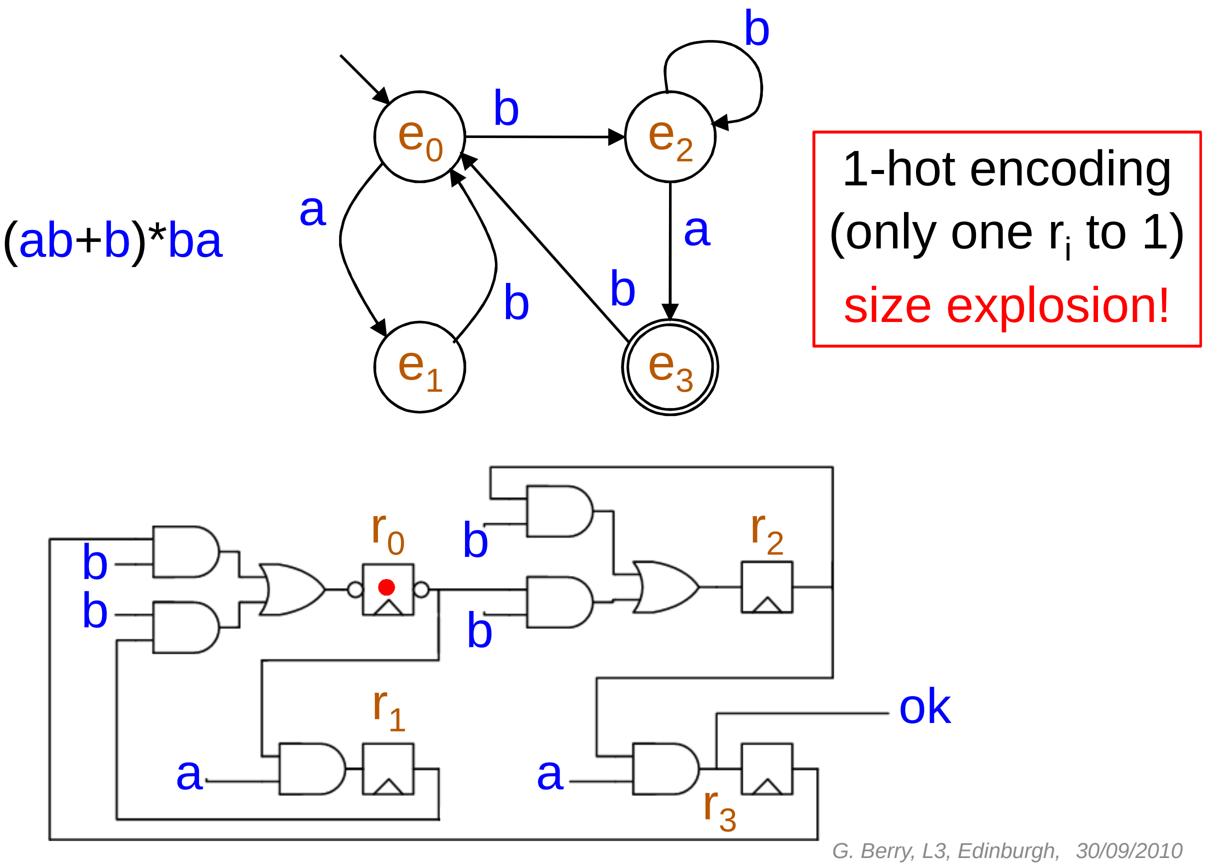
Agenda

1. Derivatives and Translation to Automata
2. The Berry-Sethi Algorithm
3. Determinization and minimization
4. From Automata to Boolean Circuits
5. Automatic Sequences and Transcendental Numbers

Implementation by Boolean Circuits

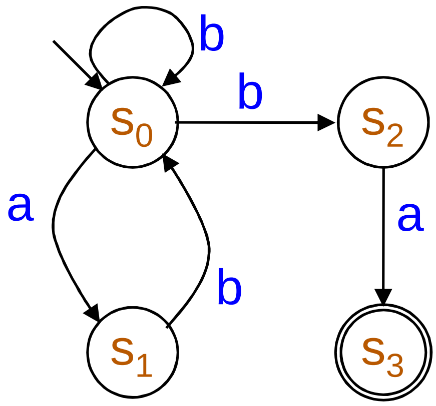


The Deterministic Case

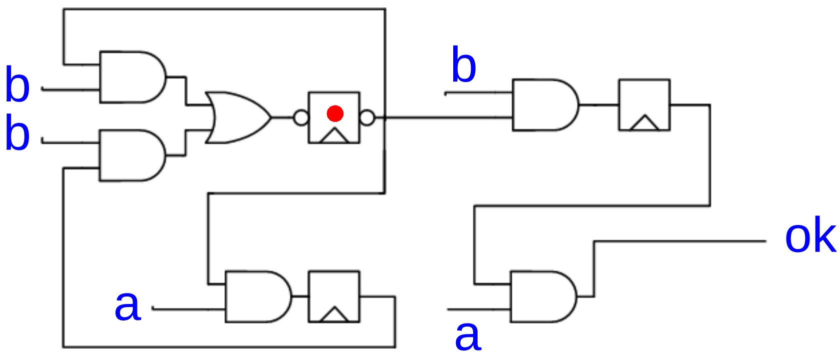


The Non-Deterministic Case

(ab+b)*ba

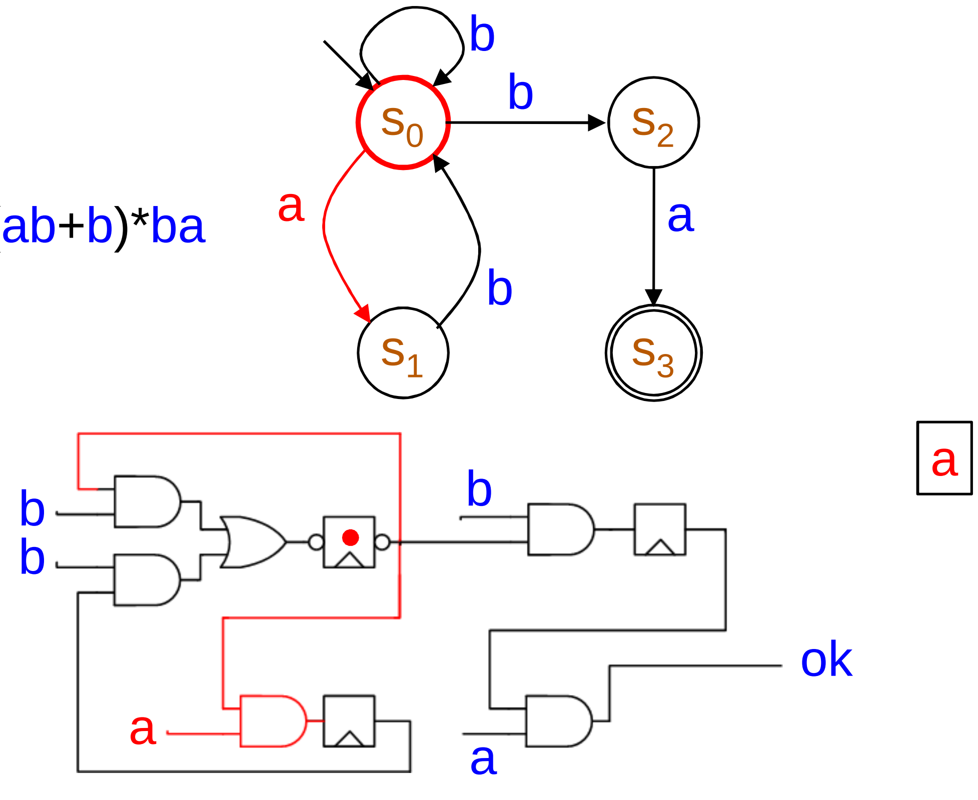


no size explosion
⇒ much better!

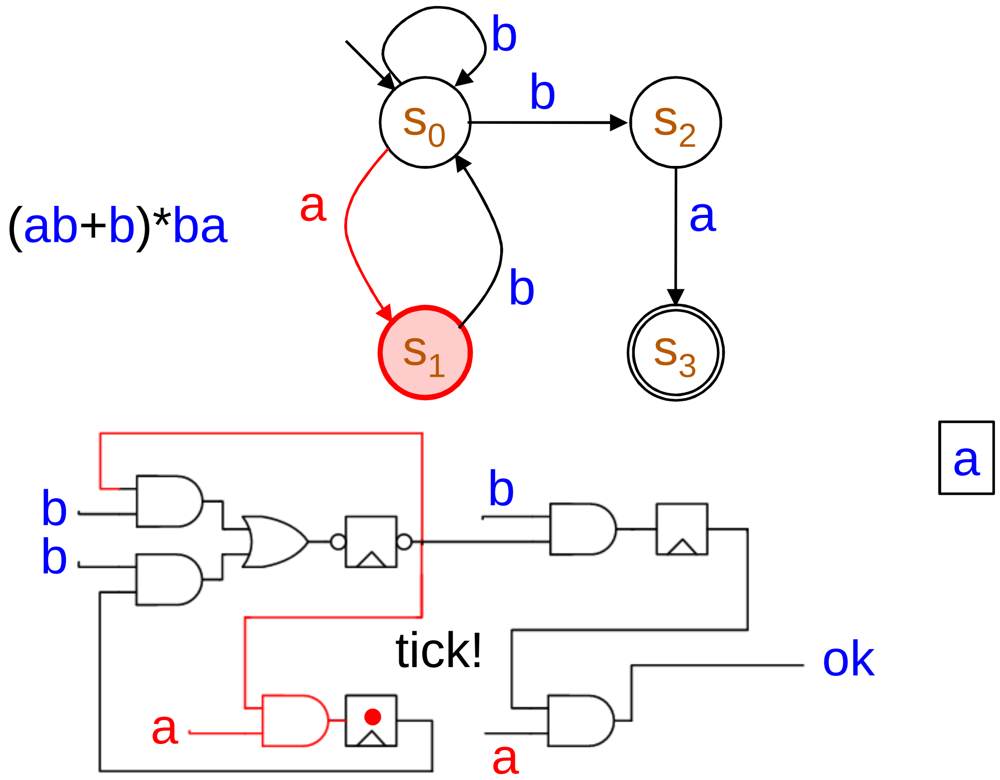


On-The-Fly Subset Construction

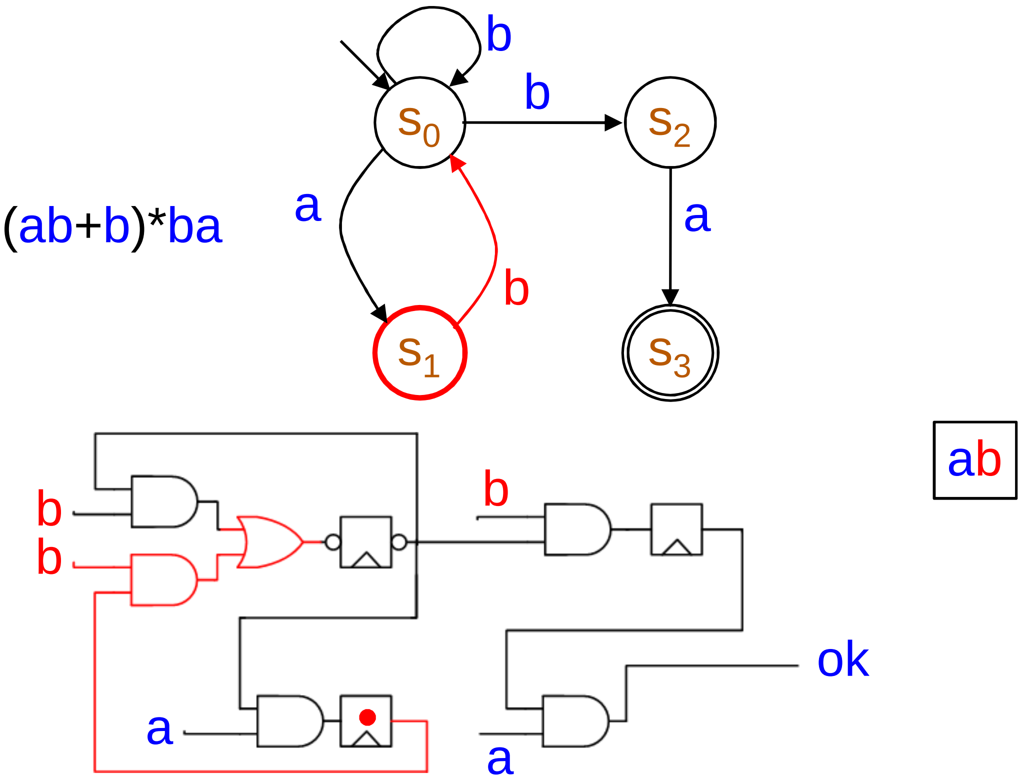
(ab+b)*ba



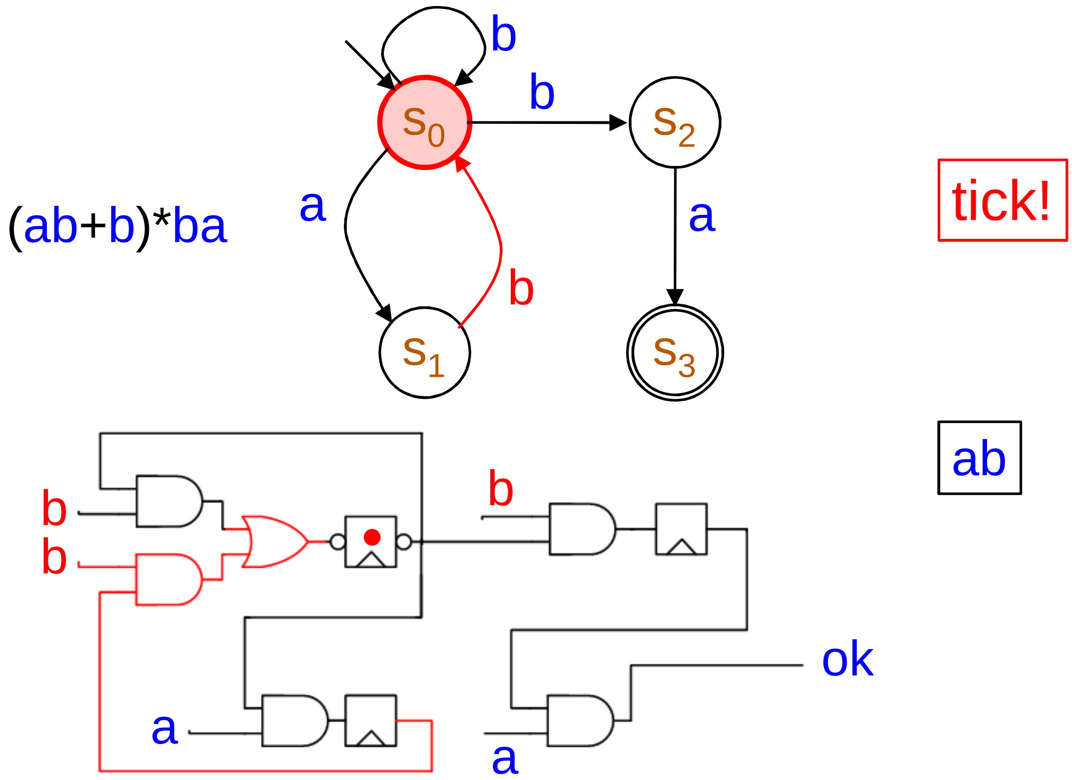
On-The-Fly Subset Construction



On-The-Fly Subset Construction

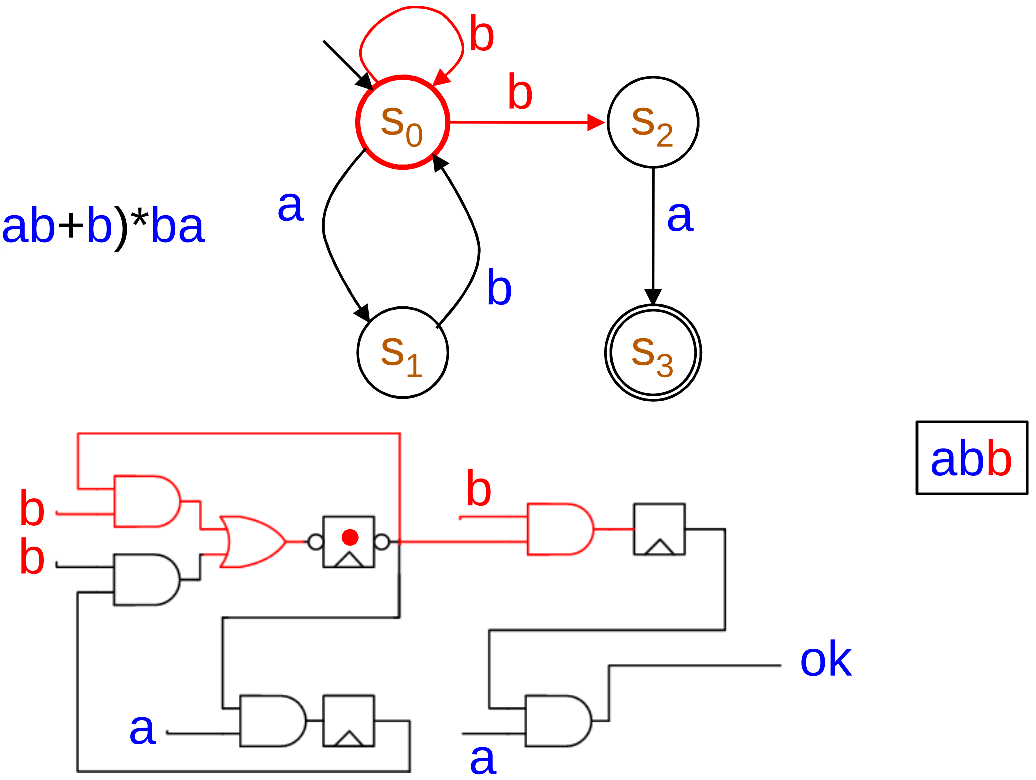


On-The-Fly Subset Construction



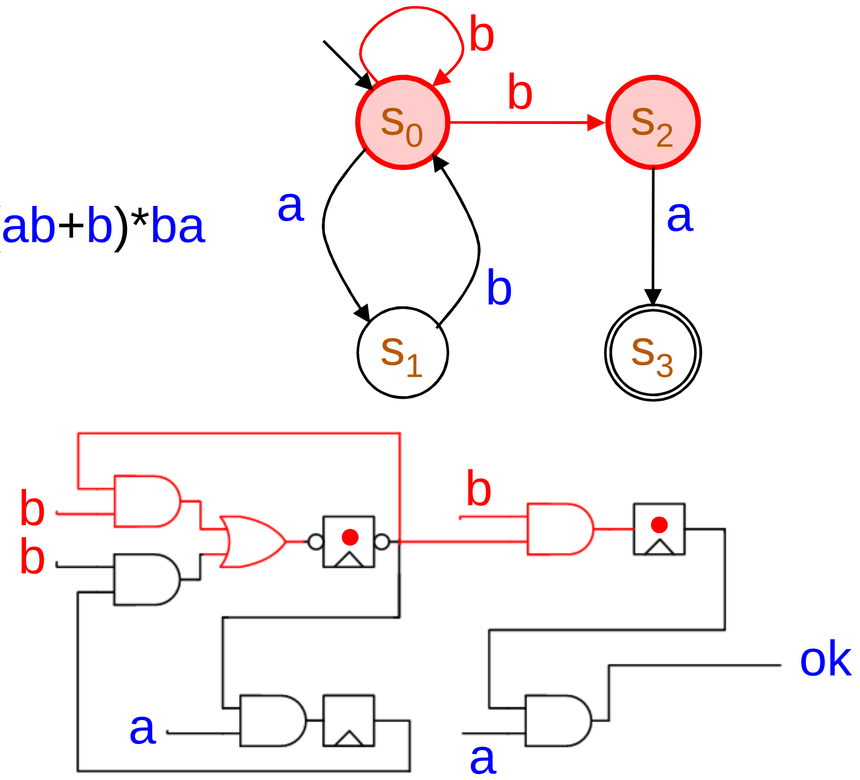
On-The-Fly Subset Construction

(ab+b)*ba



On-The-Fly Subset Construction

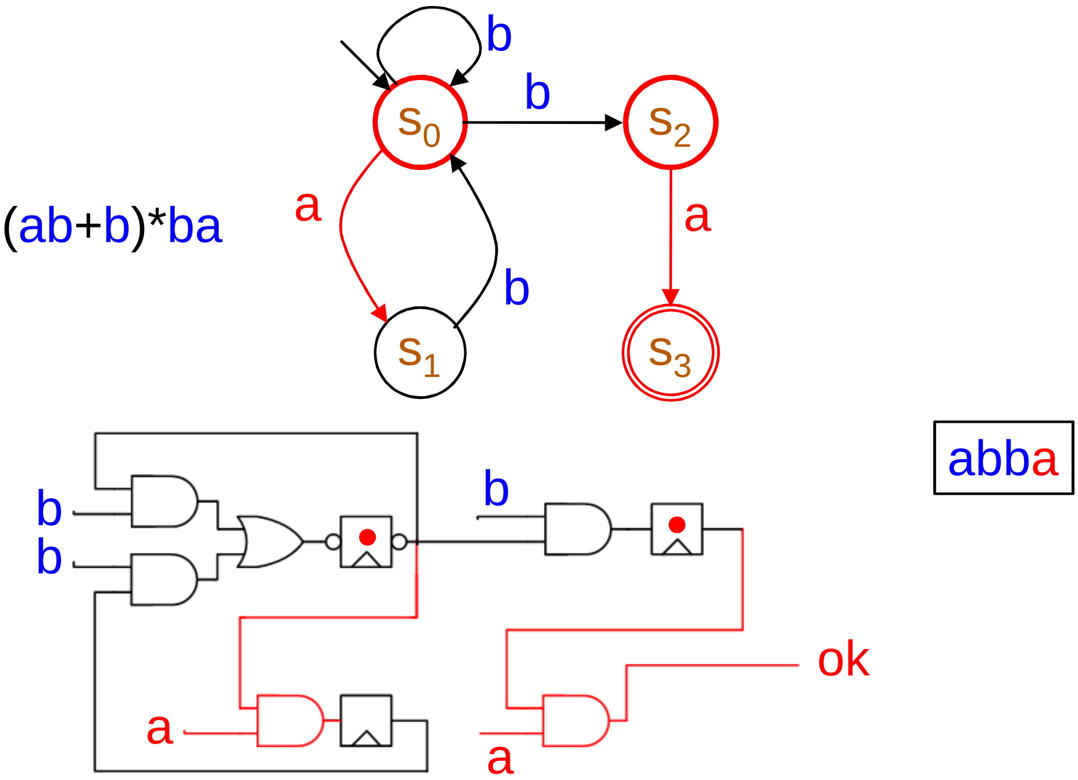
(ab+b)*ba



tick!

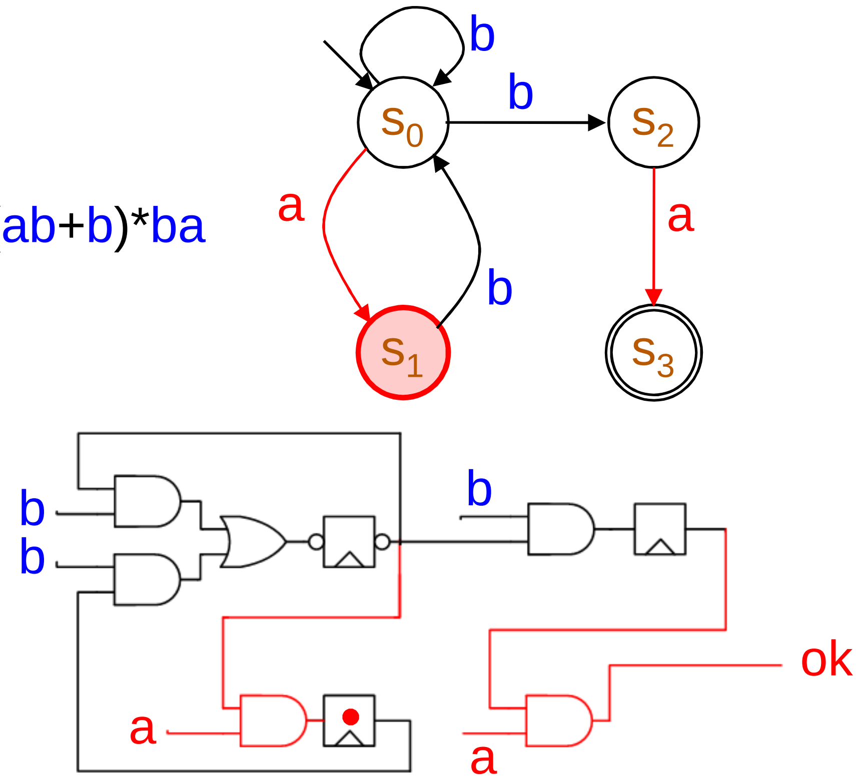
abb

On-The-Fly Subset Construction



On-The-Fly Subset Construction

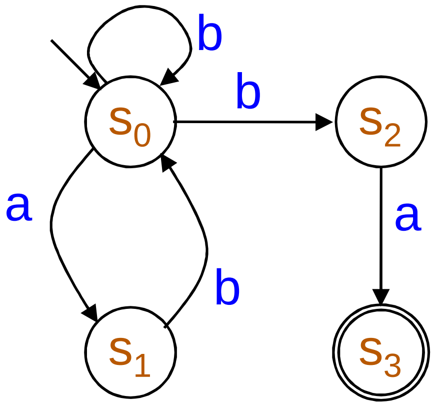
(ab+b)*ba



tick!

abba

Esterel v7 implementation




```
module Autom :
input a, b ;
output ok;
local {r0, r1, r2} : reg;
refine r0 : init true;
sustain {
   next r0 <= (r0 or r1) and b,
   next r1 <= r0 and a,
   next r2 <= r0 and b,
   ok <= r2 and a }
end module
```

Linear compiling into C, C++, VHDL, Verilog, etc.

Fundamental Practical Result

Any regular expression of size n is recognized by a circuit with $n+1$ registers and at most n^2 gates

- Scales up in size, is always superfast
- Almost always better than determinization
- The circuit can be cleverly optimized (see Synchronous Languages appendix)

Agenda

1. Regular Expressions
2. Derivatives and Translation to Automata
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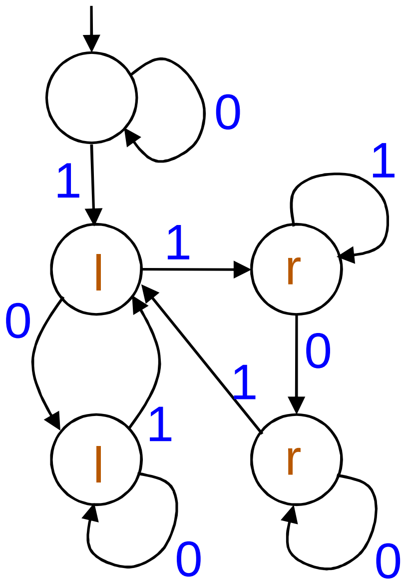
Paper-Folding Automatic Sequence

- Fold a paper, always on the same size
- Unfold it and list 0 / 1 direction changes
- Play the automaton from left to right

	1000 : l
1 : l	1001 : l
10 : l	1010 : l
11 : r	1011 : r
100 : l	1100 : r
101 : l	1101 : l
110 : r	1110 : r
111 : r	1111 : r



automatic sequence

Paper Folding Transcendence

- From an automatic sequence s_i, build the real number
  $0,s_1s_2...s_n...$
- Theorem : this number is either rational or transcendtal

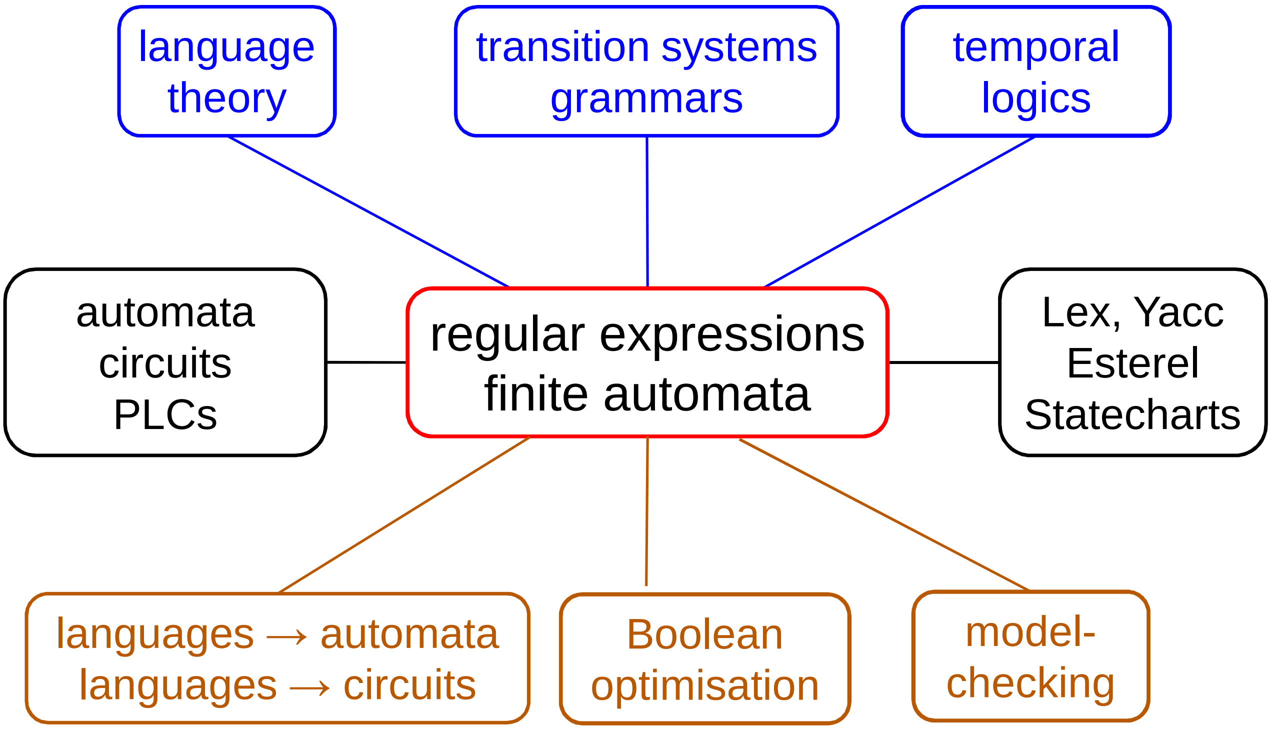
	1000 : l
1 : l	1001 : l
10 : l	1010 : l
11 : r	1011 : r
100 : l	1100 : r
101 : l	1101 : l
110 : r	1110 : r
111 : r	1111 : r

l=1, r=0

PF = 0,1101100111001001...
is transcendental

cf. Allouche, Mendès-France, Rauzy, etc.

The Finite-State Systems Map



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